

§ 向量場奇點的標數

1. 切線轉角 (Hopf Umlaufssatz) 定理
2. Holonomy 角 $\delta_c \alpha = \int_{\Omega} K dM$, $Ind X = \frac{1}{2\pi} \delta_c \varphi$
3. Gauss-Bonnet 定理
4. Hopf-Poincare 定理 奇點標數和 = $\chi(M)$
5. $\sum_{p_i} Ind_{p_i} X = \frac{1}{2\pi} \int_M K dM = \chi(M)$

Definition 3.1 A point $p \in M$ is said to be a **singular point** of X if $X_p = 0$. A singular point is said to be an **isolated singularity** if there exists a neighborhood $V \subset M$ of p such that p is the only singular point of X in V .

Since M is compact, if all the singularities of X are isolated then they are in finite number (as otherwise they would accumulate on a non-isolated singularity).

To each isolated singularity $p \in V$ of $X \in \mathfrak{X}(M)$ one can associate an integer number, called the **index** of X at p , as follows:

- (i) fix a Riemannian metric in M ;
- (ii) choose a positively oriented orthonormal frame $\{F_1, F_2\}$, defined on $V \setminus \{p\}$, such that

$$F_1 = \frac{X}{\|X\|},$$

let $\{\bar{\omega}^1, \bar{\omega}^2\}$ be the dual coframe and let $\bar{\omega}_1^2$ be the corresponding connection form;

- (iii) possibly shrinking V , choose a positively oriented orthonormal frame $\{E_1, E_2\}$, defined on V , with dual coframe $\{\omega^1, \omega^2\}$ and connection form ω_1^2 ;
- (iv) take a neighborhood D of p in V , homeomorphic to a disc, with a smooth boundary ∂D , endowed with the induced orientation, and define the index I_p of X at p as

$$2\pi I_p = \int_{\partial D} \sigma,$$

where $\sigma := \bar{\omega}_1^2 - \omega_1^2$ is the form in Proposition 2.7.

Recall that σ satisfies $\sigma = d\theta$, where θ is the angle between E_1 and F_1 . Therefore I_p must be an integer. Intuitively, the index of a vector field X measures the number of times that X rotates as one goes around the singularity anticlockwise, counted positively if X itself rotates anticlockwise, and negatively otherwise.

Example 3.2 In $M = \mathbb{R}^2$ the following vector fields have isolated singularities at the origin with the indicated indices (cf. Fig. 4.1):

- (1) $X_{(x,y)} = (x, y)$ has index 1;
- (2) $Y_{(x,y)} = (-y, x)$ has index 1;
- (3) $Z_{(x,y)} = (y, x)$ has index -1;
- (4) $W_{(x,y)} = (x, -y)$ has index -1.

Theorem 3.3 (Gauss–Bonnet) *Let M be a compact, oriented, 2-dimensional manifold and let X be a vector field in M with isolated singularities p_1, \dots, p_k . Then*

$$\int_M K = 2\pi \sum_{i=1}^k I_{p_i} \quad (4.7)$$

for any Riemannian metric on M , where K is the Gauss curvature.

例 求 $X=(x^3-3xy^2, y^3-3x^2y)$ 在 $(0,0)$ 的標數 do Carmo p.283

$I = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{dt} dt$, 其中 φ 是向量場 v 與 X_u 的夾角

$$v = (x^3 - 3xy^2, y^3 - 3x^2y), \quad x = \cos t, y = \sin t$$

$$|v|^2 = (\cos^3 t - 3\cos t \sin^2 t)^2 + (\sin^3 t - 3\cos^2 t \sin t)^2 = \cos^6 t + \sin^6 t + 3\sin^2 t \cos^2 t = \dots = 1$$

$$|v| = 1, \text{ 所以設 } \cos^3 t - 3\cos t \sin^2 t = \cos \varphi, \sin^3 t - 3\cos^2 t \sin t = \sin \varphi$$

則

$$-\sin \varphi \frac{d\varphi}{dt} = -3\sin t \cos^2 t + 3\sin^3 t - 6\sin t \cos^2 t = -3\sin t(3\cos^2 t - \sin^2 t) \dots (1)$$

$$\cos \varphi \frac{d\varphi}{dt} = 3\sin^2 t \cos t - 3\cos^3 t + 6\sin^2 t \cos t = 3\cos t(3\sin^2 t - \cos^2 t) \dots (2)$$

$$\left(\frac{d\varphi}{dt}\right)^2 = 9\sin^2 t(9\cos^4 t + \sin^4 t - 6\sin^2 t \cos^2 t)$$

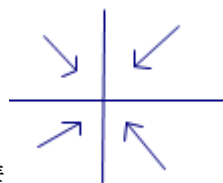
$$+ 9\cos^2 t(9\sin^4 t + \cos^4 t - 6\sin^2 t \cos^2 t)$$

$$= 81\sin^2 t \cos^2 t + 9(\sin^6 t + \cos^6 t) - 54\sin^2 t \cos^2 t$$

$$= 27\sin^2 t \cos^2 t + 9(1 - 3\sin^2 t \cos^2 t) = 9$$

所以 $I = \pm 3$

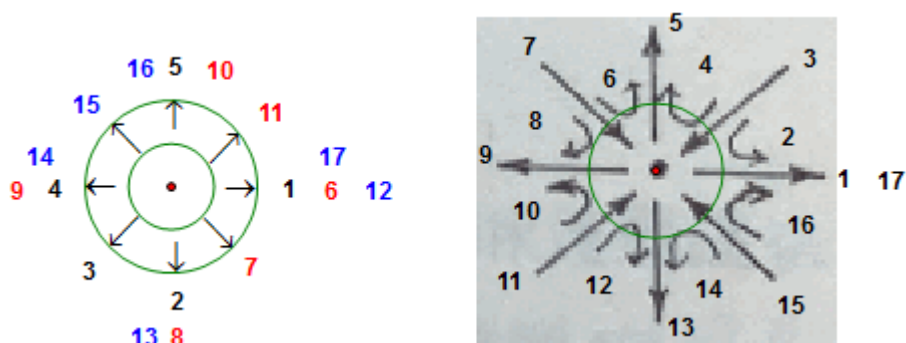
$$V_{(1,1)} = (-2, -2), V_{(-1,1)} = (2, -4), V_{(-1,-1)} = (2, 2), V_{(1,-1)} = (-2, 4)$$



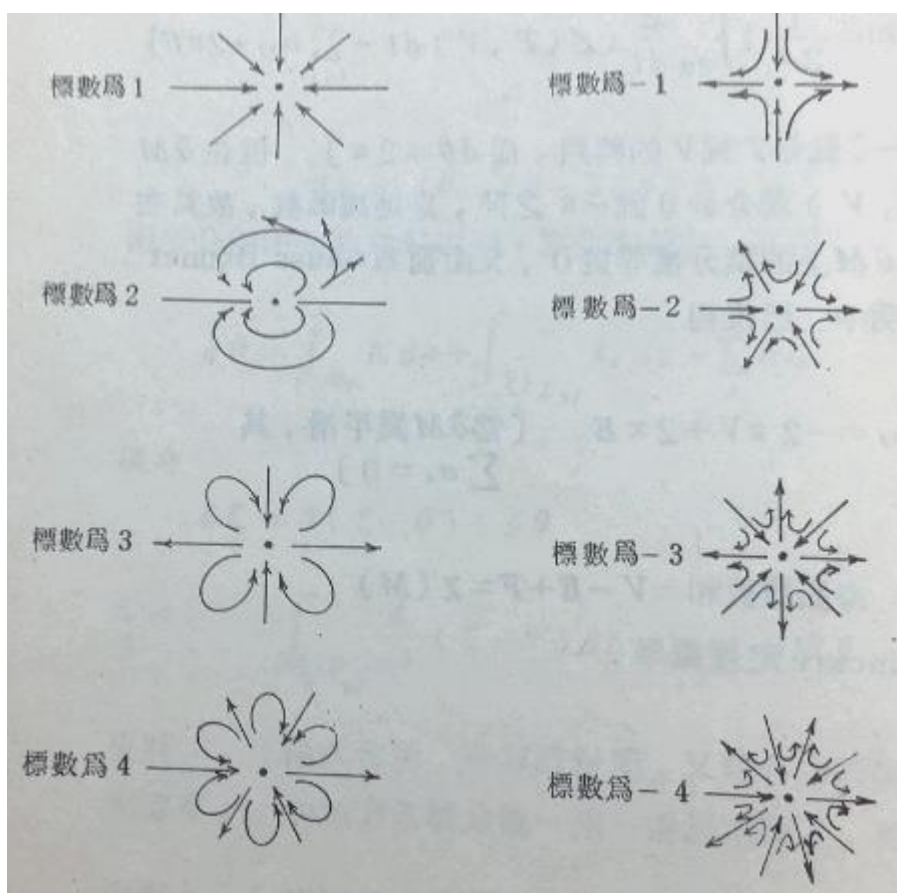
圖長這樣

, 所以 $I = -3$,

所以原向量場的水流圖就是下圖



繞奇點逆時針轉一圈，水流的切向量順時針轉 3 圈，所以 $I = -3$ 奇點標數與附近水流的情況。



Theorem 3.3 (Gauss–Bonnet) *Let M be a compact, oriented, 2-dimensional manifold and let X be a vector field in M with isolated singularities p_1, \dots, p_k . Then*

$$\int_M K = 2\pi \sum_{i=1}^k I_{p_i} \tag{4.7}$$

for any Riemannian metric on M , where K is the Gauss curvature.

The prove is in p.144

1. 初等微分幾何講稿 黃武雄 p.151 Hopf-Poincare 標數定理
2. 大域微分幾何 p.30
3. An Introduction to Riemannian Geometry p.142 Gauss-Bonnet theorem
4. Differential Geometry of Curves and Surface P.do Carmo P.282 Exercise 4-5 (6)