§ hyperbolic plane $H=\{(x,y)\,|\,y>0, x,y\in R\}$ with metric $ds^2=\frac{1}{y^2}(dx^2+dy^2)$，

稱為 Poincare 半平面模型。

一. $A(x_0,y_1), B(x_0,y_2)$ 求 $\overline{AB}=$

Length of arc $\Delta s=\int_{y_1}^{y_2}\sqrt{g_{ij}\frac{dx^i}{dy}\frac{dx^j}{dy}}dy=\int_{y_1}^{y_2}\frac{dy}{y}=\ln\frac{y_2}{y_1}$

二. geodesics

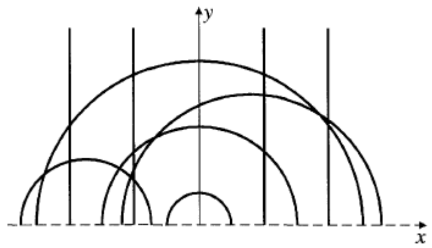

FIGURE 3.8 The upper half plane with a negatively curved metric. Geodesics are semicircles and straight lines that intersect the x-axis vertically.

Compute $\Gamma^k_{ij}=\frac{1}{2}\sum_l g^{kl}\left(\frac{\partial g_{jl}}{\partial x^i}+\frac{\partial g_{il}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^l}\right)$

$$g_{ij}=\begin{pmatrix}\frac{1}{y^2} & 0\\ 0 & \frac{1}{y^2}\end{pmatrix}, g^{ij}=\begin{pmatrix}y^2 & 0\\ 0 & y^2\end{pmatrix}$$

$$\Gamma^1_{12}=\Gamma^x_{xy}=\frac{1}{2}g^{1l}\left(\frac{\partial g_{2l}}{\partial x}+\frac{\partial g_{1l}}{\partial y}-\frac{\partial g_{12}}{\partial x^l}\right)=\frac{1}{2}g^{11}\left(\frac{\partial g_{21}}{\partial x}+\frac{\partial g_{11}}{\partial y}-\frac{\partial g_{12}}{\partial x}\right)=-\frac{1}{y}$$

So $\Gamma^1_{12}=\Gamma^1_{21}=-\frac{1}{y}, \Gamma^2_{11}=\frac{1}{y}, \Gamma^2_{22}=-\frac{1}{y}$，and the others =0

Geodesics satisfy $\ddot{x}^i+\sum_{j,k}\Gamma^i_{jk}\dot{x}^j\dot{x}^k=0$

We have $\ddot{x}-\frac{2}{y}\dot{x}\dot{y}=0$ and $\ddot{y}+\frac{1}{y}(\dot{x}^2-\dot{y}^2)=0$

$$\text{i.e.}\begin{cases}\frac{d^2x}{ds^2}-\frac{2}{y}\frac{dx}{ds}\frac{dy}{ds}=0\text{........(1)}\\ \frac{d^2y}{ds^2}+\frac{1}{y}\left(\left(\frac{dx}{ds}\right)^2-\left(\frac{dy}{ds}\right)^2\right)=0\text{...(2)}\end{cases}$$

Consider $\frac{d}{ds}(y\frac{dy}{ds}(\frac{dx}{ds})^{-1}+x)$

$$\frac{d}{ds}(y\frac{dy}{ds}(\frac{dx}{ds})^{-1}+x)=\frac{dy}{ds}\cdot\frac{dy}{ds}\cdot(\frac{dx}{ds})^{-1}+y\frac{d^2y}{ds^2}(\frac{dx}{ds})^{-1}+y\frac{dy}{ds}(\frac{-\frac{d^2x}{ds^2}}{(\frac{dx}{ds})^2})+\frac{dx}{ds}$$

$$=(\frac{dx}{ds})^{-2}\left\{(\frac{dy}{ds})^2\frac{dx}{ds}+y\cdot\frac{d^2y}{ds^2}\cdot\frac{dx}{ds}-y\cdot\frac{dy}{ds}\cdot\frac{d^2x}{ds^2}+(\frac{dx}{ds})^3\right\}=0$$

Substitude $\frac{d^2x}{ds^2},\frac{d^2y}{ds^2}$ from (1) (2)

Then $y\frac{dy}{ds}(\frac{dx}{ds})^{-1}+x=x_0$, $y\frac{dy}{ds}+x\frac{dx}{ds}=x_0\frac{dx}{ds}$, $y\cdot\frac{dy}{ds}+(x-x_0)\frac{ds}{ds}=0$

$\therefore (x-x_0)^2+y^2=l^2$ is the geodesics of H 。

三. Compute the Gaussian curvature by Cartan structure equations

Cartan formula :

$$d\omega^i=\sum_j\omega^j\wedge\omega_j^i \text{ , } \omega_i^j+\omega_j^i=0 \text{ , } d\omega_i^j=\Omega_i^j+\sum\omega_i^k\wedge\omega_k^j$$

$X_1=\frac{\partial}{\partial x},X_2=\frac{\partial}{\partial y}$ then $<X_1,X_1>=\frac{1}{y^2},<X_1,X_2>=0,<X_2,X_2>=\frac{1}{y^2}$

Take $E_1=yX_1,E_2=yX_2$ as the orthonormal frames

Then $\omega^1=\frac{1}{y}dx,\omega^2=\frac{1}{y}dy$

$$d\omega^1=(-\frac{1}{y^2})dy\wedge dx=\frac{1}{y^2}dx\wedge dy=\omega^1\wedge\omega^2$$

$$d\omega^2=0$$

By $d\omega^i=\sum_j\omega^j\wedge\omega_j^i$

$$d\omega^1=\omega^1\wedge\omega_1^1+\omega^2\wedge\omega_2^1=\omega^2\wedge\omega_2^1=\omega_1^2\wedge\omega^2$$

$$d\omega^2=\omega^1\wedge\omega_1^2+\omega^2\wedge\omega_2^2=\omega^1\wedge\omega_1^2=0$$

Let $\omega_1^2=a\omega^1+b\omega^2$, $\omega^1\wedge\omega_1^2=0$ so b=0

$\omega_1^2 \wedge \omega^2 = d\omega^1 = \omega^1 \wedge \omega^2 = a\omega^1 \wedge \omega^2$ so a=1

$d\omega_1^2 = d\omega^1 = \omega^1 \wedge \omega^2 = -K\omega^1 \wedge \omega^2$ then K=-1

$$\omega = \begin{pmatrix} 0 & -\frac{1}{y} \\ \frac{1}{y} & 0 \end{pmatrix} dx \text{，} d\theta = -\omega \wedge \theta \text{，} d\omega + \omega \wedge \omega = \Omega$$

$$\Omega = d\omega = \begin{pmatrix} 0 & \frac{1}{y^2} \\ -\frac{1}{y^2} & 0 \end{pmatrix} dy \wedge dx = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \theta^1 \wedge \theta^2$$

The Gauss curvature is $K = \frac{1}{|g|} g(\Omega(\partial_x, \partial_y)\partial_y, \partial_x) = y^4(-\frac{1}{y^4}) = -1$

Or $K = \Omega_2^1(E_1, E_2) = -\frac{1}{y^2}(dx \wedge dy)(y\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}) = -(dx \wedge dy)(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = -1$

四. Hyperbolic plane is a Lie group

一個李群 G 是一個平滑流形(smooth maniflod) 同時是一個群，使得群的運算

$G \times G \to G \qquad G \to G$

$(g,h) \mapsto gh \qquad g \mapsto g^{-1}$ 都是可微映射

我們把 $H = \{(x,y) \in R^2 \,|\, y > 0\}$ 上的每一點與可逆仿射映射(affine map)

$h: R \to R, h(t) = yt + x$ 等同(identify)，所有這樣的映射所成的集合在結合律為一群。

因此我們在 H 上引入(induce)一個群結構。

Exercise 1.7.17 (3)

(a) Show that the induced group operation is given by $(x,y)\bullet(z,w) = (yz + x, yw)$ and that H，with this group operation is a Lie group。

(b) Show that the derivative of left translation map $L_{(x,y)}: H \to H$ at a point $(z,w) \in H$ is represented in the above coordinates by the matrix

$$(dL_{(x,y)})_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \text{。}$$

Conclude that the left-invariant vector field $X^V \in \chi(H)$ determined by the vector $V=\xi\frac{\partial}{\partial x}+\varsigma\frac{\partial}{\partial y}\in\eta\equiv T_{(0,1)}H,(\xi,\varsigma\in R)$ is given by $X^V_{(x,y)}=\xi y\frac{\partial}{\partial x}+\varsigma y\frac{\partial}{\partial y}$

(c) Given $V,W\in\eta$ compute [V , W]

(d) Determine the flow of the vector field X^V，and give an expression for the exponential map $\exp:\eta\to H$

(e) Confirm your results by first showing that H is the subgroup of GL(2) firmed by the matrices $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ with y>0

解

(a)

Given two affine maps g(t)=yt+x and h(t)=wt+z，we have

$(g\circ h)(t)=$g(h(t))=g(w t +z)=yw t + yz+x

Therefore the group operation is given by $(x,y)\bullet(z,w)=(yz+x,yw)$

The identity element is e=(0,1)，hence

$$(z,w)=(x,y)^{-1}\Leftrightarrow(yz+x,yw)=(0,1)\Leftrightarrow(z,w)=(-\frac{x}{y},\frac{1}{y})$$

Therefore the maps $(g,h)\to g\bullet h$ and $g\to g^{-1}$ are smooth hence H is a Lie group。

(b)

Because $L_{(x,y)}(z,w)=(yz+x,yw)$，the matrix representaion of

$(dL_{(x,y)})(z,w)=\begin{pmatrix} y & 0 \\ o & y \end{pmatrix}$，therefor $X^V_{(x,y)}$ has components $\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}=\begin{pmatrix} y\xi \\ y\eta \end{pmatrix}$

(c)

If $V=\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y},W=\varsigma\frac{\partial}{\partial x}+\omega\frac{\partial}{\partial y}$，then

$$[X^V,X^W]=[\xi y\frac{\partial}{\partial x}+\eta y\frac{\partial}{\partial y},\varsigma y\frac{\partial}{\partial x}+\omega y\frac{\partial}{\partial y}]=(\eta\varsigma-\omega\xi)y\frac{\partial}{\partial x}$$

Therefore $[V,W]=[X^V,X^W]_{(0,1)}=(\eta\varsigma-\omega\xi)\frac{\partial}{\partial x}$

(d)

The flow of X^V is given by the solution of the system of ODEs

$$\begin{cases} \dot{x} = \xi y \\ \dot{y} = \eta y \end{cases}$$

Which is $\begin{cases} x = x_0 + \dfrac{y_0 \xi (e^{\eta t} - 1)}{\eta} \\ y = y_0 e^{\eta t} \end{cases}$ for $\eta \neq 0$

And $\begin{cases} x = x_0 + y_0 \xi t \\ y = y_0 \end{cases}$ for $\eta = 0$

The exponential map is obtained by setting $(x_0, y_0) = e = (0,1)$

and t=1： $\exp(V) = \begin{pmatrix} \dfrac{\xi(e^{\eta} - 1)}{\eta} & e^{\eta} \end{pmatrix}$ for $\eta \neq 0$

and $\exp(V) = (\xi, 1)$ for $\eta = 0$

(e) The multiplication of two such matrices is

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} yw & yz + x \\ 0 & 1 \end{pmatrix},$$

which reproduces the group operation on H. Therefore H can be identified with the corresponding subgroup of $GL(2)$. A curve $c : (-\varepsilon, \varepsilon) \to H$ with $c(0) = I$ is then given by

$c(t) = \begin{pmatrix} y(t) & x(t) \\ 0 & 1 \end{pmatrix}$ with x(0)=0 and y(0)=1，and its derivative at t=0 is

$$\dot{c}(0) = \begin{pmatrix} \dot{y}(0) & \dot{x}(0) \\ 0 & 0 \end{pmatrix}$$

We conclude that $\mathfrak{h}$ can be identified with the vector space of matrices of the form

$$\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix}.$$

The Lie bracket must then be given by

$$\left[\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \omega & \zeta \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega & \zeta \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \omega & \zeta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \eta\zeta - \omega\xi \\ 0 & 0 \end{pmatrix},$$

which agrees with (c). Moreover, the exponential map must be given by

$$\exp\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix} = \sum_{k=0}^{+\infty} \frac{1}{k!}\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix}^k$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \eta^2 & \eta\xi \\ 0 & 0 \end{pmatrix} + \cdots,$$

yielding

$$\exp\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^\eta & \frac{\xi(e^\eta - 1)}{\eta} \\ 0 & 1 \end{pmatrix}$$

for $\eta \neq 0$ and

$$\exp\begin{pmatrix} \eta & \xi \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix},$$

for $\eta = 0$, which agrees with (d).

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§ left-invariant metric

The hyperbolic plane corresponds to the left-invariant metric $g = \frac{1}{y^2}(dx^2 + dy^2)$ on H。

The geodesics are therefor determined by the Hamiltonain function $K : T^*H \to R$ given by $K(x, y, p_x, p_y) = \frac{y^2}{2}(p_x^2 + p_y^2)$

(a) Determine the lift to T^*H of the action of H on itself by left translation, and check that it preserves the Hamiltonian K.

(b) Show that the functions

$$F(x, y, p_x, p_y) = y p_x \quad \text{and} \quad G(x, y, p_x, p_y) = y p_y$$

are also H-invariant, and use this to obtain the quotient Poisson structure on T^*H/H. Is this a symplectic manifold?

(c) Write an expression for the momentum map for the action of H on T^*H, and use it to obtain a nontrivial first integral I of the geodesic equations. Show that the projection on H of a geodesic for which $K = E$, $p_x = l$ and $I = m$ satisfies the equation

$$l^2x^2 + l^2y^2 - 2lmx + m^2 = 2E.$$

Assuming $l \neq 0$, what are these curves?

解

(a) From the expression of the group operation it is clear that

$$(x, y)^{-1} = \left(-\frac{x}{y}, \frac{1}{y}\right),$$

and so

$$L_{(a,b)^{-1}}(x, y) = \left(\frac{x}{b} - \frac{a}{b}, \frac{y}{b}\right).$$

Therefore, by Example 5.4, the lift of the action of H on itself to T^*H is given by

$$\begin{aligned}(a, b) \cdot (p_x dx + p_y dy) &= \left(L_{(a,b)^{-1}}\right)^* (p_x dx + p_y dy) \\ &= \frac{p_x}{b} dx + \frac{p_y}{b} dy,\end{aligned}$$

which can be written in local coordinates as

$$(a, b) \cdot (x, y, p_x, p_y) = \left(bx + a, by, \frac{p_x}{b}, \frac{p_y}{b}\right).$$

Since

$$K\left(bx + a, by, \frac{p_x}{b}, \frac{p_y}{b}\right) = \frac{b^2 y^2}{2}\left(\frac{{p_x}^2}{b^2} + \frac{p_y^2}{b^2}\right) = K(x, y, p_x, p_y),$$

we see that K is H-invariant.

(b) The functions F and G are H-invariant as

$$F\left(bx + a, by, \frac{p_x}{b}, \frac{p_y}{b}\right) = by\frac{p_x}{b} = yp_x = F(x, y, p_x, p_y)$$

and

$$G\left(bx + a, by, \frac{p_x}{b}, \frac{p_y}{b}\right) = by\frac{p_y}{b} = yp_y = G(x, y, p_x, p_y).$$

These functions are coordinates on the quotient manifold T^*H/H (they are the components on a left-invariant basis), and so the Poisson structure of the quotient is determined by

$$\{F, G\} = X_F \cdot G = \frac{\partial F}{\partial p_x}\frac{\partial G}{\partial x} + \frac{\partial F}{\partial p_y}\frac{\partial G}{\partial y} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p_x} - \frac{\partial F}{\partial y}\frac{\partial G}{\partial p_y}$$
$$= -p_x y = -F.$$

The Poisson bivector on the quotient is therefore

$$B = \{F, G\}\frac{\partial}{\partial F} \otimes \frac{\partial}{\partial G} + \{G, F\}\frac{\partial}{\partial G} \otimes \frac{\partial}{\partial F}$$
$$= -F\frac{\partial}{\partial F} \otimes \frac{\partial}{\partial G} + F\frac{\partial}{\partial G} \otimes \frac{\partial}{\partial F}.$$

Since B vanishes for $F = 0$, the quotient T^*H/H is not a symplectic manifold.

(c) Differentiating the expression

$$L_{(a,b)}(x, y) = (bx + a, by)$$

along a curve $(a(t), b(t))$ through the identity $e = (0, 1)$, it is readily seen that the infinitesimal action of $V = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} \in \mathfrak{h}$ is

$$X^V = (\alpha + \beta x)\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y}.$$

From Example 5.4, the momentum map for the action of H on T^*H is the map $J : T^*H \to \mathfrak{h}^*$ given by

$$J(p_x dx + p_y dy)(V) = (p_x dx + p_y dy)(X^V) = (\alpha + \beta x)p_x + \beta y p_y.$$

Since K is H-invariant, J is constant along the Hamiltonian flow of K, and so, choosing $\alpha = 0$ and $\beta = 1$, we obtain the nontrivial first integral

$$I(x, y, p_x, p_y) = xp_x + yp_y$$

for the Hamiltonian flow of K (in addition to the obvious first integrals K and p_x). A geodesic for which $K = E$, $p_x = l$ and $I = m$ then satisfies

$$y^2\left({p_x}^2 + p_y^2\right) = 2E \Leftrightarrow y^2 l^2 + (m - xl)^2 = 2E,$$

which for $l \neq 0$ is the equation of a circle centered on the x-axis.

Exercses

1. Let H be the upper half plane $\{(x,y)\in R^2 \,|\, y>0\}$ 。For any $\alpha\in R$ ，define the metric $g_\alpha = \frac{1}{y^\alpha}(dx^2+dy^2)$

   (a) If $\alpha \neq 2$ ，prove that (H, g_α) is incomplete

   (b) Write (x,y) as z=x+iy。For any (a,b,c,d)$\in R^4$ with ad-bc=1，show that $z \mapsto \frac{az+b}{cz+d}$ define an isometry of (H, g_2)

   (c) S^1 is the circle $\{e^{i\theta}\}$

   Consider the following metric on $H\times S^1$

   $$g = \frac{dx^2+dy^2}{y^2} + (d\theta + \frac{1}{y}dx)^2$$

   Deote $y\partial_x - \partial_\theta$ by e_1 ，$y\partial_y$ by e_2 and ∂_θ by e_3

   Calculate its curvature R_{2112}, R_{3113} and R_{3223}

   Where $R_{jiij} = <(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i} - \nabla_{[e_i,e_j]})e_i, e_j>$

2. Let M be a hyperbolic manifold。Suppose $\gamma_0 : S^2 \to M$ is a closed geodesic，whose γ_0' has constant length。Is it possible to find a one-parameter family of closed curves with $\gamma : S^1 \times \{t\in R : -\varepsilon < t < \varepsilon\} \to M$

   with $\gamma(\cdot,0) = \gamma_0(\cdot)$ and $\frac{\partial \gamma}{\partial t}|_{t=0} \perp \gamma_0'$ everywhere on γ_0

   such that $\frac{d}{dt}|_{t=0} L[\gamma(\cdot,t)] < 0$? Give your reason。

   Here ，$L[\gamma(\cdot,t)]$ means the arc length of the closed curve $\gamma(\cdot,t) : S^1 \to M$

3.

$R^x_{yxy} = -y^{-2}$

$R_{xx} = -y^2, R_{xy} = 0, R_{yy} = -y^{-2}$

Curvature scalar $R = -\frac{2}{a^2}$