§ Flows

1. A flow of a vector field

$\varphi : U \to M \quad \varphi_t(p) = \gamma(t), t \in (-\varepsilon, \varepsilon)$

$\dfrac{d(\varphi_t(p))}{dt} = \dfrac{d\gamma}{dt} = X$ ，則稱 φ_t 是向量場 X 的 flow

φ_t 是 1-parametergroup， $\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q)$ ， $\varphi_0 = id$ entity

一個向量場 X，其 local flow 定義一個 one-parameter group of diffeomorphism 則稱 X 為完備。

例 $X = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$ ， $Y = z\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial z}$ 求 X 的 flow φ_t

$$\frac{d\varphi_t(p)}{dt} = X_{\varphi_t(p)}, \varphi_{0(p)} = p$$

$$\begin{cases} \dot{\varphi}_t^1(p) = X^1(\varphi_{t(p)}) = 0 \\ \dot{\varphi}_t^2(p) = X^2(\varphi_{t(p)}) = -\varphi_t^3 \\ \dot{\varphi}_t^3(p) = X^3(\varphi_{t(p)}) = \varphi_{t(p)}^2 \end{cases} \Rightarrow \varphi_t^1 = c, \ddot{\varphi}_t^2 = -\dot{\varphi}_t^3 = -\varphi_t^2$$

Then $\varphi_t^1 = C,\ \varphi_t^2 = A\cos t + B\sin t, \varphi_t^3 = A\sin t - B\cos t$

A，B，C are function of p=(x y z)

And $\varphi_0(x,y,z) = (x,y,z)$ ，so C=x，A=y，B=-z

所以 $\varphi_t(x,y,z) = (x, y\cos t - z\sin t, y\sin t + z\cos t)$

X 是繞 x 軸旋轉的向量場，是 Killing vector field。

V.I.Arnold

M phase space：position velocity

曲線 $x = \varphi(t)$ ， $\dfrac{d\varphi}{dt} = v(t,\varphi(t))$ ， $\dot{x} = v(t,x)$ 或者簡寫成 $\dot{x} = v(x)$

For $v(x) = \dfrac{d}{dt}\Big|_{t=0}(g^t x)$ ，the phase velocity vector field，the phase flow is the one-parameter diffeomorphism group。

例 $\dot{x} = kx$

$\dot{\varphi} = k\varphi$ with $\varphi_0 = id$ 解出 $\varphi = xe^{kt}$，So the phase flow is the group $\{xe^{kt}\}$

例1. Find the phase flow of (1) $\dot{x} = 1$ (2) $\dot{x} = x - 1$ (3) $\dot{x} = \sin x, o < x < \pi$

(2) $\dot{\varphi}_t(x) = \varphi_t(x) - 1$，$\varphi_0(x) = x$

$\varphi_t(x) = e^{t+c} + 1$，$\varphi_0(x) = e^c + 1 = x$，$e^c = x - 1$

所以 $g^t x = \varphi_t(x) = (x-1)e^t + 1$

(3) 查表得知 $\int \csc x dx = -\ln|\csc x + \cot x| + c = -\ln\left|\cot\frac{x}{2}\right| + c$

$\dot{\varphi} = \sin\varphi$，$\frac{d\varphi}{\sin\varphi} = dt$ 兩邊積分

$-\ln\left|\cot\frac{\varphi}{2}\right| = t + c$，$\cot\frac{\varphi}{2} = e^{-t} + c$ by $\varphi_0(x) = x$

$\varphi_t(x) = 2\,\text{arccot}(e^{-t} \times \tan\frac{x}{2})$

例2. Find the phase flows of the systems

(1) $\begin{cases}\dot{x} = y \\ \dot{y} = 0\end{cases}$ (2) $\begin{cases}\dot{x} = y \\ \dot{y} = 1\end{cases}$ (3) $\begin{cases}\dot{x} = \sin y \\ \dot{y} = 0\end{cases}$

(2) $\dot{\varphi}_t^2 = 1$，$\varphi_t^2 = y + t$ for $\varphi_0(x, y) = (x, y)$

$\dot{\varphi}_t^1 = \varphi_t^2 = y + t$，得 $\varphi_t^1 = ty + \frac{1}{2}t^2 + x$

所以 $g^t(x, y) = (\varphi_t^1, \varphi_t^2) = (x + ty + \frac{1}{2}t^2, y + t)$

(3) $g^t(x, y) = (x + t\sin y, y)$

Q：是否每一個 smooth vector field 是一個 flow 的 phase velocity vector field？

例 $v(x) = x^2$

$\dot{\varphi}_t = \varphi_t^2, \frac{d\varphi}{\varphi^2} = dt$ with $\varphi_0(x) = x$ 解出 $g^t x = \varphi_t(x) = \frac{x}{1 - xt}$

容易驗證 $\varphi_t^x \circ \varphi_s^x = \ldots = \varphi_{t+s}^x$

當$t \neq 0$，$g^t x$在$x = \frac{1}{t}$沒有定義

所以 v(x)沒有 phase flow

用向量場的說法是

$X = x^2 \frac{d}{dx}$，求$\dot{x} = x^2$的積分曲線 with initial condition $x(0) = x_0 \neq 0$

則$x(t) = \frac{x_0}{1 - t x_0}$在$t = \frac{1}{x_0}$沒有定義

$\varphi_t(x) = \frac{x}{1 - tx}$ 所以 X 非完備。

所以 Arnold 問：

是否每一個 smooth vector field 是一個 flow 的 phase velocity vector field？

即 是否每一個量場皆完備，答案當然是否定的。

2. Euler-Lagrange flows

$L: TM \to R$ Lagrangian，$u: [0,1] \to M$

The action of L，$A(u) = \int_0^1 L(u(t), \dot{u}(t)) dt$

考慮 A 的變分，u is a critical point $\Leftrightarrow$

$$\frac{\partial L}{\partial u}(u, \dot{u}) - \frac{d}{dt}\left(\frac{\partial L}{\partial v}(u, \dot{u})\right) = 0 \text{，} v = \dot{u} \quad \text{(Euler-Lagrange equation)}$$

If M is compact，the extremals(critical point) of A give rise to a complete flow

$\phi_t: TM \to TM$ called the Euler-Lagrange flow of the Lagrangian。

The Euler-Lagrange equations for a hyper-regular Lagrangian L define a flow on M。This flow is carried by the Legendre transformation to the flow defined on T^*M by the Hamilton equations

$$\begin{cases} \dot{x}^i = \dfrac{\partial H}{\partial p_i} \ldots.(1') \\ \dot{p}_i = -\dfrac{\partial H}{\partial x^i} \ldots(2') \end{cases}$$

3. The flow of a Hamilton equations

The Hamilton equations are the equations for the flow of the vector field X_H satisfying

$$i_{X_H}\omega = -dH$$

Hamiltonian flows preserve their generating functions。i.e. $X_F F = 0$

Hamiltonian flows preserve the canonical symplectic form。

If $\varphi_t : T^*M \to T^*M$ is a Hamiltonian flow then $\varphi^*\omega = \omega$

Liouville theorem

Hamiltonian flows preserve the integral with respect to the symplectic volume form。

$\varphi_t : T^*M \to T^*M$ is a Hamiltonian flow and $F \in C^\infty(T^*M)$ is a compactly supported function then $\int_{T^*M} F \circ \varphi_t = \int_{T^*M} F$

4. Geodesic flow

M is a complete Riemannian manifold

$\gamma_{(x,v)}(t)$ is the unique geodesic with $\begin{cases} \gamma_{(x,v)}(0) = x \\ \dot{\gamma}_{(x,v)}(0) = v \end{cases}$

TM is the tangent bundle。

$\phi_t : TM \to TM$, $\phi_t(x,v) := (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))$ is a diffeomorphism

Then $\phi_{t=0}(x,v) = (x,v) = identity$

Then $\{\phi_t\}$ is a flow，with $\phi_{t+s} = \phi_t \circ \phi_s$

$SM = \{v \mid v \in TM, |v| = 1\}$

∵ geodesic travel with constant speed，ϕ_t leaves SM invariant。

That is，given $(x,v) \in SM$ for all then $\phi_t(x,v) \in SM$ 。

The restriction of ϕ_t to SM is called the geodesic flow of g。

(質點沿 geodesic 走 不受力 加速度=0 速度是常數。)

5. Ricci flow

考慮在 $M^n \times [0,T]$ 上的 Ricci flow $\frac{\partial g}{\partial t} = -2Ric(g)$

例

If $Ric(g_0) = \lambda g_0$，λ is a constant。Then a solution g(t) of $\frac{\partial g}{\partial t} = -2Ric(g)$ with $g(0) = g_0$ is given by $g(t) = (1-2\lambda t)g_0$

In particularly， for (S^n, g_0)，we have $Ric(g_0) = (n-1)g_0$，so the evolution is $g(t) = (1-2(n-1)t)g_0$ and the sphere collapses to a point at time $T = \frac{1}{2(n-1)}$

6. A flow of a Killing vector field

c.f.
[RG1101vectorfield01] [DEflows]