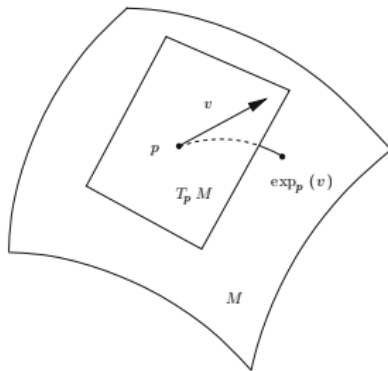§ Exponential Map

There are two kinds of exponential map：

The exponential map of a connection and a Lie group。

When a Lie group has a bi-invariant Riemannian metric，as all compact Lie groups do，
the exponential map for the Lie group coincides with the exponential map of the
Riemannian connection。

§ The exponential map of a connection，denoted as Exp_p

The exponential map

M is a manifold with an affine connection

$\gamma_V(t)$ 是從 $p = \gamma_V(0)$ 出發、切於 V 的測地

線 (即 $\gamma_V(0) = p, \dfrac{d\gamma_V}{dt}(0) = V$)

給定 $p \in M$ ，定義 $\exp_p : T_pM \to M$ 使得

$Exp_p V = \gamma_V(1), \forall V \in T_pM$

(可以導出 $\gamma_X(t) = Exp_p(tX), \forall t$)

That is defined locally。If the exponential map well-defined at every point of the tangent
bundle，then the affine connection called complete。

The range can fail to be all of M simply because there can be two points that are not
connected by any geodesic。

The domain can fail to be all of T_p because a geodesic may run into a singularity。

Manifolds that have such singularities are known as geodesically incomplete。

[An Introduction to Riemannian Geometry p.110]有 normal neighborhood 的命題，然後
定義 normal ball   p.115 習作 3.4.8 有 normal coordinates 的定義。

§ normal neighborhood

$Exp_p : T_p M \to M$ is a local diffeomorphism

$X \to \gamma(1)$ where X is the tangent vector of the geodesic γ at p。

$X = \dfrac{d\gamma}{dt}(0), Exp_p(0) = \gamma(0) = p$

Then $\exists$ V(opn) in $T_p M$，such that $Exp_p : V \to U$ is a diffeomorphism，U is called

a normal neighborhood of p。

§ Normal coodinates

Fix a point p in a Riemannian manifold M。

There is a neighborhood V of 0 in $T_p M$ and a neighborhood U of p in M such that the

exponetial map $Exp_p : V \to U$ is a diffeomorphism。

Using the exponential map we can tramsfer coordinates on $T_p M$ to M。

Choose an orthonormal basis $e_1,...,e_n$ for $T_p M$ and let $r^1,...,r^n$ be the coordinates

with respect to the orthonormal basis $e_1,...,e_n$ on $T_p M$ 。

Then $x^1 := r^1 \circ Exp_p^{-1},...,x^n := r^n \circ Exp_p^{-1}$ is a coordinates system on U such that the

tangent vectors $\dfrac{\partial}{\partial x^1},...,\dfrac{\partial}{\partial x^n}$ are orthonormal at p。

The coordinate neighborhood $(U, x^1,...,x^n)$ is called a normal neighborhood of p and

$x^1,...,x^n$ are called normal coordinates on U。

In a normal neighborhood of p，the geodesics throught p have a particularly simple

expression，for the coordinate expression for the geodesic $\gamma(t) = Exp_p(at)$ for

$a = \sum a^i e_i \in T_p M$ is $x(\gamma(t)) = r \circ Exp_p^{-1}(\gamma(t) = at$

We write this as $(x^1,...,x^n) = (a^1 t,...,a^n t)$

Exercse 3.4.8

(2) Let M be a Riemannian manifold and ∇ the Levi–Civita connection on M. Given $p \in M$ and a basis $\{v_1, \ldots, v_n\}$ for T_pM, we consider the parameterization $\varphi : U \subset \mathbb{R}^n \to M$ of a normal neighborhood given by

$$\varphi(x^1, \ldots, x^n) = \exp_p(x^1 v_1 + \cdots + x^n v_n)$$

(the local coordinates $(x^1, \ldots, x^n)$ are called **normal coordinates**). Show that:

(a) in these coordinates, $\Gamma^i_{jk}(p) = 0$ (**Hint:** Consider the geodesic equation);
(b) if $\{v_1, \ldots, v_n\}$ is an orthonormal basis then $g_{ij}(p) = \delta_{ij}$.

指數映射用來定義常態座標(Riemannian normal coordinates，or locally inertial coordinates)以簡化計算。

§ Theorem

In a normal neighborhood $(U, x^1, \ldots, x^n)$ of p，all the partial derivatives of g_{ij} and all

the Christoffel symbols Γ^i_{jk} vanish at p。

Let $(x^1, \ldots, x^n) = (a^1 t, \ldots, a^n t)$ be a geodesic through p。

It satisfies the geodesic equations $\ddot{x}^i + \sum_{j,k} \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$ or $\sum \Gamma^i_{jk} a^j a^k = 0$

Since this is true for all $(a^1, \ldots a^n)$ at p。Setting $(a^1, \ldots a^n) = (0, \ldots 1, 0, \ldots 0, 1, \ldots 0)$

with $a^j = a^k = 1$ and all other entries 0，we get $\Gamma^i_{jk} + \Gamma^i_{kj} = 0$

by the symmetry of the connection，$\Gamma^i_{jk} = 0$ at p。(At other points，not all of

$(x^1, \ldots, x^n) = (a^1 t, \ldots, a^n t)$ will be geodesics。)

by the compatibiity of the connection ∇ with the metric，

$$\partial_k g_{ij} = \partial_k <\partial_i, \partial_j> = <\nabla_{\partial_k}\partial_i, \partial_j> + <\partial_i, \nabla_{\partial_k}\partial_j>$$

At p，$\nabla_{\partial_k} \partial_i = 0$ and $\nabla_{\partial_k} \partial_j = 0$ since all the Christofell symbols vanish。

Therefore，$(\partial_k g_{ij})(p) = 0$

Normal coordinates are especially useful for computution，because at the point p，all

$\nabla_{\partial_k} \partial_i = 0$

在一個黎曼流形(M,g) 因為存在 totally normal neighborhoods，因此可以用極小化的測地線連接 M 上距離夠小的兩點，因此而有 Hopf-Rinow 定理。

Gauss lemma

在 normal neighborhood 內 其邊界 $\partial B_p(r)$ 與從 p 點出發的測地線垂直。

證明在 p.111~112

§ Left-invariant vector field on a Lie group

$\eta = T_e G$ is the Lie algebra of Lie group G

$$TM \overset{df}{\to} TN \qquad (f_* X)_{f(p)} := (df)_p X_p，\text{the map is a diffeomorphism}$$
$$X \uparrow \qquad \uparrow f_* X$$
$$M \overset{f}{\to} N \qquad \begin{aligned} &L_g : G \to G \\ & \quad h \mapsto g \cdot h \end{aligned} \quad \text{Is called left-muptiplication}$$

For a induced map $(L_g)_* : T_h(G) \to T_{gh}(G)$ ，a vector field X is called left-invariant

$\Leftrightarrow (L_g)_* X = X$ for all $g \in G$，that is $((L_g)_* X)_{gh} = X_{gh}$

or $(dL_g)_h X_h = X_{gh}$ for all $g, h \in G$

let { left-invariant vector fields on G }$= \chi_L(G)$ ， then $\chi_L(G) \cong \eta$

for each $V \in \eta$ ， $X_g^V := (dL_g)_e V$ for any $g \in G$

Then the vector field X_h^V is left-invariant

$$(\ dL_g)_h X_h^V = (dL_g)_h (dL_h)_e V = (d(L_g \circ L_h))_e V = (dL_{gh})_e V = X_{gh}^V$$

§ The local flow φ_t of a left-invariant vector field X on a Lie group G commutes with

left-multiplication $L_g \circ \varphi_t = \varphi_t \circ L_g$ for all $g \in G$ ， whenever both sides are defined 。

§ The exponential map of a Lie group ， denoted as exp

Let G is a Lie group ， $X \in \eta$ 。 Let X_γ be the integral curve of X starting at the

identity 。 Then the exponential map $\eta \to G$ ， $\exp X = \gamma_X(1)$

The exponential map $\exp : \eta \to G$ is the map that ， to each $V \in \eta$ assigns the value $\psi_1(e)$ ， where ψ_t is the flow of the left-invariant vector field X^V

If G is a group of matrices ， then for $A \in \eta$ $\exp A = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

$h(t) = e^{At}$ satisfies (1) $h(0) = e^0 = I$ (2) $\dfrac{dh}{dt} = e^{At} A = h(t)A$

h(t) is the flow of X^A at the identity and so $\exp A = \psi_1(e) = e^A$

Proposition 15.9. (i) *For $X_e \in \mathfrak{g}$, the integral curve starting at e of the left-invariant vector field X is* $\exp(tX_e) = c_X(t) = \varphi_t(e)$.

(ii) *For $X_e \in \mathfrak{g}$ and $g \in G$, the integral curve starting at g of the left-invariant vector field X is* $g \exp(tX_e)$.

(iii) *For $s, t \in \mathbb{R}$ and $X_e \in \mathfrak{g}$,* $\exp\big((s+t)X_e\big) = (\exp s X_e)(\exp t X_e)$.

(iv) *The exponential map* $\exp : \mathfrak{g} \to G$ *is a C^∞ map.*

(v) *The differential at 0 of the exponential map,* $\exp_{*,0} : T_0(\mathfrak{g}) = \mathfrak{g} \to T_e G = \mathfrak{g}$ *is the identity map.*

(vi) *For the general linear group* $\mathrm{GL}(n, \mathbb{R})$,

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \quad \text{for } A \in \mathfrak{gl}(n, \mathbb{R}).$$

(proof followed) 杜武亮 p.120

Let $\phi : R \to G$ be a smooth Lie group homomorphism，and let $A = \phi^{'}(0) \in T_e G$ be a tangent vector at the identity。

(R 是加法群，G 是乘法群 $\therefore \phi(0) = I$)

Such homomorphism is called an one-parameter subgroup of G。

Let G=GL(n, R) this a matrix group，so the tangent vector at identity is a matrix。

Since $\phi(s+t) = \phi(s)\phi(t)$ ，evaluating the derivative at s=0 gives

($\dfrac{d\phi}{dt} = \lim_{h \to 0} \dfrac{\phi(t+h) - \phi(t)}{h} = \lim_{h \to 0} \dfrac{\phi(t)\phi(h) - \phi(t)}{h} = \phi(t) \lim_{h \to 0} \dfrac{\phi(h) - I}{h} = \phi'(0)\phi(t)$)

$\dfrac{d}{dt} \phi(t) = \phi'(0) \cdot \phi(t) = A\phi(t)$ ，with $\phi(0) = I$

Then $\phi(t) = e^{At}$

So for any Lie group G，we define the exponential map $\exp : \eta \to G$

$\exp(X) = \gamma_X(1)$ is a diffeomorphism

Let $A \in T_e G$ ，and $\phi : R \to G$ be the unique homomorphism such that $\phi'(0) = A$

Then $\exp(A) = e^A = \phi(1)$

The map $t \to e^{tA}$ is a local diffeomorphism from $T_e G$ to G。

例 G=SO(3, R) then　$T_e G = so(3, R)$

$$J_X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_Y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, J_Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\exp : so(3, R) \rightarrow SO(3, R)$

$$\exp(tJ_X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$　rotation around x-axis by angle t

$$\exp(tJ_Y) = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix}$$　rotation around y-axis by angle t

$$\exp(tJ_Z) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$　rotation around z-axis by angle t

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