2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$ form a subset of R^4 。

Is it a compact subset？

Is SL(2,R) compact？

1. Closedness

The determinant map

$$f : \mathbb{R}^4 \to \mathbb{R}, \quad f(a,b,c,d) = ad - bc$$

is continuous. The set $SL(2,\mathbb{R})$ is the preimage of the closed set $\{1\}$ under f:

$$SL(2,\mathbb{R}) = f^{-1}(\{1\}).$$

Since the preimage of a closed set under a continuous function is closed, $SL(2,\mathbb{R})$ is a **closed** subset of $\mathbb{R}^4$.

2. Boundedness

Consider the matrix:

$$\begin{pmatrix} t & 0 \\ 0 & \frac{1}{t} \end{pmatrix}$$

for $t \in \mathbb{R} \setminus \{0\}$. The determinant is $t \cdot \frac{1}{t} = 1$, so these matrices belong to $SL(2,\mathbb{R})$. As $t \to \infty$, the entries become unbounded. This shows that $SL(2,\mathbb{R})$ is **not bounded**.

SO(2) is compact。

1. **Closedness:**
   - The conditions defining $SO(2)$ are:

$$A^T A = I \quad \text{and} \quad \det(A) = 1.$$

   - These conditions are given by polynomial equations in the entries of A. The set of solutions to such polynomial equations is closed in $\mathbb{R}^4$.

2. **Boundedness:**
   - Each matrix in $SO(2)$ has entries that are sines and cosines of angles θ, which are always between -1 and 1.
   - Hence, the set is bounded.