§ Let (M, g) be a connected Riemannian manifold，with $\dim M \geq 3$

(a) State the second Bianchi identity for the Riemann crvature

(b) Suppose that its Ricci curvature is proportional to the metric tensor。Namely，thre exists $f \in C^{\infty}(M;R)$ such that Ric(X ,Y)=f(p)g(X, Y) for any $p \in M$，and $X, Y \in T_p M$。

Prove that f must be a contstant function。

We are given Ric(X, Y)=f(p)g(X,Y)，$\forall p \in M, X, Y \in T_p M$，where $f \in C^{\infty}(M, R)$

We want to prove that f is a constant function。

The second Bianchi idetity for the Riemannian curvature：

$(\nabla_X Ric)(Y, Z) + \nabla_Y Ric(Z, X) + (\nabla_Z Ric)(X, Y) = 0$

Since Ric=fg，we have $\nabla_W Ric(X, Y) = \nabla_W (fg(X, Y))$

Because the Levi-Civita connection is compatible with the metric ($\nabla g = 0$)，we get：

∇_WRic(X,Y)=($\nabla_W f$)(X,Y)

Thus $\nabla_Z Ric(X, Y) = (\nabla_Z f) g(X, Y)$

Plugging $\nabla Ric = (\nabla f) \otimes g$ into the Bianchi identity gives：

$(\nabla_Z f) g(X, Y) + (\nabla_X f) g(Y, Z) + (\nabla_Y f) g(Z, X) = 0$

Fix a point $p \in M$。Let z=X and Y be orthogonal to X，then

$(\nabla_X f) g(X, Y) + (\nabla_X f) g(Y, X) + (\nabla_Y f) g(X, X) = 0$

Since $g(Y, X) = 0$，this reduce to $(\nabla_Y f) g(X, X) = 0$

But $g(X, X) \neq 0$ for a non zero X，so $\nabla_Y f = 0$ for all $Y \perp X$

We can conclude that $\nabla f = 0$ at every point $p \in M$

Since ∇f=0 everywhere，this means the gradient of f vanishes globally。On a **connected manifold**，any smooth function with zero gradient must be **constant**。

Thus f=constant on M。

1. The condition Ric=fg means that the Ricci curvature looks the same in all directions at each point, up to a scaling by f。
2. The Ricci tensor measures how volumes deviate from Euclidean volumes under the manifold's geometry. If this deviation is uniformly proportional to the metric, it suggests a very **symmetric curvature structure.**
3. The **second Bianchi identity** imposes strong compatibility conditions on how curvature can vary。 In dimensions ≥3，these conditions force the scaling factor f to be the **same everywhere,**，reflecting **global geometric rigidity**。

§ Connection to Einstein manifolds

Our result shows that any manifold with Ricci curvature proportional to the metric is automatically an Einstein manifold，provided dim(M)≥3。

§ Role in General Relativity

- The **Einstein field equations** (without matter) are:

$$\mathrm{Ric} - \frac{1}{2}Rg + \Lambda g = 0,$$

where R is the scalar curvature and Λ is the cosmological constant.

- If we assume $\mathrm{Ric} = fg$, then comparing with Einstein's equations shows that f must be constant—tying our purely geometric result directly to **physical constraints** on spacetime geometry.

§ Implications in Ricci flow and geometric analysis

- The **Ricci flow** evolves a metric $g(t)$ according to:

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}.$$

- A fixed point of the Ricci flow satisfies $\mathrm{Ric} = \lambda g$, i.e., it's an Einstein metric.
- Our result shows that steady-state solutions under certain curvature conditions must have constant proportionality, simplifying the study of long-time behavior of Ricci flows.
- This insight is essential in results like **Perelman's proof** of the **Poincaré conjecture** and the **Geometrization conjecture**.

The result is a beautiful example of how local curvature conditions combined with global topology (connectedness of M) can enforce global geometric properties。