Consider the smooth Riemannian 2-manifold $D=\{(x,y)\in R^2; x^2+y^2<1\}$ with a metric

$\frac{dx^2+dy^2}{g^2(x,y)}$ 。Here $g(x,y)=\frac{1}{2}[1-(x^2+y^2)]$

(1) Find the frame field $\{E_1,E_2\}$ and dual coframe field $\{\theta_1,\theta_2\}$

(2) Show that the connection 1-form $\omega_{12}=g_y\theta_1-g_x\theta_2$

(3) Show that the Gaussian curvature K=-1

(4) Let Δ be a geodesic triangle in D。Show that $area(\Delta)=\pi-(i_1+i_2+i_3)$ ，here

i_1,i_2,i_3 are the interior angles at the vertices of Δ

$ds^2=\frac{dx^2+dy^2}{g^2(x,y)}=\theta_1^2+\theta_2^2$ ，where $\theta_1=\frac{dx}{g(x,y)},\theta_2=\frac{dy}{g(x,y)}$

$E_1=g(x,y)\frac{\partial}{\partial x},E_2=g(x,y)\frac{\partial}{\partial y}$ with $\theta_i(E_j)=\delta_{ij}$

For a 2D Riemannian manifold，the first Cartan structure equation is：

$$d\theta_i+\sum_j\omega_{ij}\wedge\theta_j=0$$

Where ω_{ij} are the connection 1-forms。In 2D，there is only one independent connection 1-form ω_2 because $\omega_{12}=-\omega_{21}$

$$d\theta_1=d(\frac{dx}{g})=\frac{-dg}{g^2}=-\frac{g_xdx+g_ydy}{g^2}\wedge dx=\frac{g_y}{g^2}dx\wedge dy$$

$$d\theta_2=d(\frac{dy}{g})=-\frac{g_xdx+g_ydy}{g^2}\wedge dy=-\frac{g_x}{g^2}dx\wedge dy$$

Apply Cartan first structure equation

$d\theta_1+\omega_2\wedge\theta_2=0$ $\frac{g_y}{g^2}dx\wedge dy+\omega_2\wedge\frac{dy}{g}=0$

$\omega_2=g_y\frac{dx}{g}+$(terms involving dy)

$d\theta_2-\omega_2\wedge\theta_1=0$ $\frac{g_x}{g^2}dx\wedge dy-\omega_2\wedge\frac{dx}{g}=0$ this gives $\omega_2\wedge dx=\frac{g_x}{g}dx\wedge dy$

Thus $\omega_2=-g_x\frac{dy}{g}+$(terms involving dx)

Finally，we get $\omega_2=g_y\theta_1-g_x\theta_2$

Second Cartan structure eqution $\Omega_2 = d\omega_2 = K\theta_1 \wedge \theta_2$

$$d\omega_2 = d(g_y\theta_1) - d(g_x\theta_2) = (g_{yy}dy + g_{yx}dx) \wedge \theta_1 + g_y d\theta_1 - (g_{xx}dx + g_{xy}dy) \wedge \theta_2 - g_x d\theta_2$$

Given that $g(x,y) = \frac{1}{2}[1-(x^2+y^2)]$, then

$$g_x = -x, g_y = -y, g_{xx} = -1, g_{yy} = -1, g_{xy} = g_{yx} = 0$$

$$d\omega_2 = -dy \wedge \theta_1 + dx \wedge \theta_2 + \frac{x^2+y^2}{g^2} dx \wedge dy$$

$$-dy \wedge \theta_1 + dx \wedge \theta_2 = \frac{2}{g} dx \wedge dy = \frac{2g}{g^2} dx \wedge dy$$

$$d\omega_2 = \frac{[1-(x^2+y^2)]+(x^2+y^2)}{g^2} dx \wedge dy = \frac{dx \wedge dy}{g^2}$$

Since $\theta_1 \wedge \theta_2 = \frac{dx \wedge dy}{g^2(x,y)}$, we rewrite： $d\omega_2 = -\frac{dx \wedge dy}{g^2}$ (?)

$d\omega_2 = K\theta_1 \wedge \theta_2$ thus K=-1

這是一個 hyperbolic plane

Poincare disk model

In two dimensions, with respect to these frames and the Levi-Civita connection, the connection forms are given by the unique skew-symmetric matrix of 1-forms ω that is torsion-free, i.e., that satisfies the matrix equation $0 = d\theta + \omega \wedge \theta$. Solving this equation for ω yields

$$\omega = \frac{2(y\,dx - x\,dy)}{1-|\mathbf{x}|^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the curvature matrix is

$$\Omega = d\omega + \omega \wedge \omega = d\omega + 0 = \frac{-4\,dx \wedge dy}{(1-|\mathbf{x}|^2)^2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Therefore, the curvature of the hyperbolic disk is

$$K = \Omega^1_2(e_1, e_2) = -1.$$