(1) Riemann tensor $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

(2) J(t) is a vector field J(t) along γ，satisfies $\frac{D^2 J}{dt^2} + R(J,\mathrm{T})\mathrm{T} = 0$ for all $t \in [0,l]$。

Then J(t) is called a Jacobi field。

(3) A Killing vector field

(a) $L_X g = 0$ ($L_X g_{\mu\nu} = \nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$ in local coordinates)

(b) div(K)=0

(c) In terms of the Levi-Civita connection，that is $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$ for all vectors Y and Z。Or $\nabla_\mu K^\nu = -\nabla_\nu K^\mu$ in local coordinates。

1. On a Remannian manifold (M, g)，a vector field U is called a Killing vector field if it is infinitesimally an isometry，namely $\frac{d}{dt}\big|_{t=0} \varphi_t^* g = 0$

   Where φ_t is the one-paremeter family of diffeomorphism generate by U

   (a) For a Killing vector field U，show that $g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0$ for any two vector fields X，Y

   (b) Suppose that U is a Killing vector field，and γ is a geodesic。

   Prove that $U\big|_\gamma$ is a Jacobi field

Prove

(a) A vector field U is called a Killing vector field if $\mathrm{L_U} g = 0$，in terms of the Levi-Civita connection，this condition is equivalent to $g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0$ for any two vector fields X,Y

(b) A vector field J along a geodesic γ is a Jacobi field if it satisfies the Jacobi equation：$\nabla_V \nabla_V J + R(V,J)V = 0$

Where $\mathrm{V} = \dot{\gamma}$ is the tangent vector to the geodesic。

R(X,Y)Z is the Riemann curvature tensors。

Since U is a Killing vector field，it satisfies the Killing equqtion：

$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0$，applying to X=V and Y=U，we get $\nabla_V U = -\nabla_U V$

Since γ is a geodesic，$\nabla_V V = 0$

Taking the covariant derivative along V，$\nabla_V \nabla_V U = -\nabla_V \nabla_U V$

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$ ，we substitute X=V,Y=U and Z=V to get：

$R(V,U)V = \nabla_V \nabla_U V - \nabla_U \nabla_V V = \nabla_V \nabla_U V$ since $\nabla_V V = 0$

And then $\nabla_V \nabla_V U = -R(V,U)V$

i.e. $\nabla_V \nabla_V U + R(V,U)V = 0$ This is just the Jacobi equation，meaning that U restricted to γ，denoted $U\big|_\gamma$ is a Jacobi field。

重寫一遍

Since U is a Killing field，$\nabla_V U = -\nabla_U V$ ，兩邊取∇_V得$\nabla_V \nabla_V U = -\nabla_V \nabla_U V$

$R(V,U)V = \nabla_V \nabla_U V - \nabla_U \nabla_V V = \nabla_V \nabla_U V$ (因為γ is a geodesic，$\nabla_V V = 0$)

$R(V,U)V = \nabla_V \nabla_U V = -\nabla_V \nabla_V U$

$\nabla_V \nabla_V U + R(V,U)V = 0$ 此即為 Jacobi equation，U 是一 Jacobi field，換句話說 $U\big|_\gamma$ 是一 Jacobi field。

Killing vector field 與 Jacobi field 的關係比較清楚一些了。