Differential Forms

§1 外微分 $d\omega$

$\omega = a_I dx^I$ is a k-form，then $d\omega = \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$

一. $f: R^n \to R$ is a 0-form then

$$df = \frac{\partial f}{\partial x^j} dx^j = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$ …梯度(grad) is a 1-form

二. $\omega = a_i dx^i$ is a 1-form，then

$$d\omega = (\frac{\partial a_i}{\partial x^j} dx^j) \wedge dx^i$$ is a 2-form

When n=3，$d\omega = \begin{vmatrix} dx^2 \wedge dx^3 & dx^3 \wedge dx^1 & dx^1 \wedge dx^2 \\ \frac{\partial}{\partial x^1} & \frac{\partial}{\partial x^2} & \frac{\partial}{\partial x^3} \\ a_1 & a_2 & a_3 \end{vmatrix}$ …旋度(curl)

三. $\omega = a_{ij} dx^i \wedge dx^j$ is a 2-form，then $d\omega = (\frac{\partial a_{ij}}{\partial x^k} dx^k) \wedge dx^i \wedge dx^j$

When n=3，$d\omega = (\frac{\partial a_{23}}{\partial x^1} + \frac{\partial a_{31}}{\partial x^2} + \frac{\partial a_{12}}{\partial x^3}) dx^1 \wedge dx^2 \wedge dx^3$ …散度(div)

例1. $\omega = \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$，in $R^2/\{O\}$，求 $d\omega$

It is easy to check $d\omega = 0$

If exists a 0-form φ such that $\omega = d\varphi$

Note that $\int \frac{dx}{1+x^2} = \arctan x$，then $\varphi = \arctan(\frac{y}{x})$

$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy = \omega$，$\varphi_x = \frac{-y}{x^2+y^2}, \varphi_y = \frac{x}{x^2+y^2}$

Note that $(\tan^{-1} x)' = \frac{1}{1+x^2}$，$\frac{d}{dx}(\tan^{-1}\frac{y}{x}) = \frac{1}{1+(\frac{y}{x})^2} \times (\frac{-y}{x^2}) = \frac{-y}{x^2+y^2}$

$\varphi = \tan^{-1}\frac{y}{x} + h(y)$，$\varphi_y = \frac{x}{x^2+y^2} + h'(y)$，$\therefore h'(y) = 0$

$\varphi(x,y) = \tan^{-1}\frac{y}{x} + c$ isn't defined at y-axis/{O}

So ω is closed，but not exact in $R^2/\{O\}$

Poincare lemma：

M is a contractible smooth manifold then every closed form is exact。

習作

1. Check $(Adx+Bdy+Cdz)\wedge(Pdx+Qdy+Rdz)=\begin{vmatrix} dy\wedge dz & dz\wedge dx & dx\wedge dy \\ A & B & C \\ P & Q & R \end{vmatrix}$

2. ω是 1-form，證明 $d\omega(X\wedge Y)=X\omega(Y)-Y\omega(X)-\omega([X,Y])$

前面有一個習作

若$\eta=p(dy\wedge dz)+q(dz\wedge dx)+r(dx\wedge dy)$

$X=u^1\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial y}+u^3\frac{\partial}{\partial z}$ ，$Y=v^1\frac{\partial}{\partial x}+v^2\frac{\partial}{\partial y}+v^3\frac{\partial}{\partial z}$ 則

$\eta\cdot(X\wedge Y)=\begin{vmatrix} p & q & r \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}$ ...(*)

取$\omega=hdx$ 驗證即可

則$d\omega=(\frac{\partial h}{\partial x}dx+\frac{\partial h}{\partial y}dy+\frac{\partial h}{\partial z}dz)\wedge dx=\frac{\partial h}{\partial z}dz\wedge dx-\frac{\partial h}{\partial y}dx\wedge dy$

$\omega\cdot X=hdx(u^1\frac{\partial}{\partial x}+u^2\frac{\partial}{\partial y}+u^3\frac{\partial}{\partial z})=hu^1$ ，同理$\omega\cdot Y=hv^1$

Then $X(\omega\cdot Y)=X(hv^1)=u^1(v^1\frac{\partial h}{\partial x}+h\frac{\partial v^1}{\partial x})+u^2(v^1\frac{\partial h}{\partial y}+h\frac{\partial v^1}{\partial y})+u^3(v^1\frac{\partial h}{\partial z}+h\frac{\partial v^1}{\partial z})$

同理 $Y(\omega\cdot X)=v^1(u^1\frac{\partial h}{\partial x}+h\frac{\partial u^1}{\partial x})+v^2(u^1\frac{\partial h}{\partial y}+h\frac{\partial u^1}{\partial y})+v^3(u^1\frac{\partial h}{\partial z}+h\frac{\partial u^1}{\partial z})$

$\omega\cdot[X,Y]=h(Xv^1-Yu^1)$

$$=h(u^1\frac{\partial v^1}{\partial x}+u^2\frac{\partial v^1}{\partial y}+u^3\frac{\partial v^1}{\partial z}-v^1\frac{\partial u^1}{\partial x}-v^2\frac{\partial u^1}{\partial y}-v^3\frac{\partial u^1}{\partial z})$$

(1)-(2)-(3)得

$$X(\omega\cdot Y)-Y(\omega\cdot X)-\omega\cdot[X,Y]=(u^2v^1-u^1v^2)\frac{\partial h}{\partial y}+(u^3v^1-u^1v^3)\frac{\partial h}{\partial z}$$

$$=\begin{vmatrix} 0 & \frac{\partial h}{\partial z} & -\frac{\partial h}{\partial y} \\ u^1 & u^2 & u^3 \\ v^1 & v^2 & v^3 \end{vmatrix}=d\omega\cdot(X\wedge Y)$$

$\omega = \sum_i h_i dx^i$ ， $X = \sum_i X^i \frac{\partial}{\partial x^i}, Y = \sum_i Y^i \frac{\partial}{\partial x^i}$

則 $\omega(X) = \sum_i h_i X^i, \omega(Y) = \sum_i h_i Y^i$

$[X,Y] = \sum_i (XY^i - YX^i)\frac{\partial}{\partial x^i}$ ， $X \wedge Y = \sum_{i,j}(X^i Y^j - X^j Y^i)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$

取 $\omega = hdx^k$ ，則

$\omega(X) = hX^k, \omega(Y) = hY^k$ ， $d\omega = dh \wedge dx^k = \sum_i \frac{\partial h}{\partial x^i} dx^i \wedge dx^k$

$$X\omega(Y) = \sum_i X^i \frac{\partial}{\partial x^i}(hY^k) = \sum_i X^i (h\frac{\partial Y^k}{\partial x^i} + Y^k \frac{\partial h}{\partial x^i})$$

$$Y\omega(X) = \sum_i Y^i \frac{\partial}{\partial x^i}(hX^k) = \sum_i Y^i (h\frac{\partial X^k}{\partial x^i} + X^k \frac{\partial h}{\partial x^i})$$

$$X\omega(Y) - Y\omega(X) = h\sum_i (X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i}) + \sum_i (X^i Y^k - X^k Y^i)\frac{\partial h}{\partial x^i}$$

$$= \omega[X,Y] + d\omega(X \wedge Y)$$

§2 Lie derivative of differential form ω along X

$$L_X \omega = \frac{d}{dt}(\varphi_t^* \omega)_{t=0} = \sum_j (Xh_j)dx^j + \sum_k h_k dX^k$$

1. $d(\omega + \eta) = d\omega + d\eta$
2. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega}(\omega \wedge d\eta)$
3. $d(d\omega) = 0$
4. $\varphi : R^m \to R^n$ ，then $d(\varphi^* \omega) = \varphi^*(d\omega)$
5. $L_X(d\omega) = d(L_X \omega)$
6. $L_X \omega(Y) = L_X \omega(Y) - \omega(L_X Y)$

How about the computation of $L_X g$ ，where g is the metric。For a Killing vector field X， $L_X g = 0$

§3 pull-back a form

Example

$x = r\cos\theta, y = r\sin\theta$

$dx = \cos\theta dr - r\sin\theta d\theta$

$dy = \sin\theta dr + r\cos\theta d\theta$

Then $dx \wedge dy = rdr \wedge d\theta$

$ds^2 = dr^2 + r^2 d\theta^2$ ， $(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$dA = \sqrt{\det g_{ij}}\, dr \wedge d\theta = r dr \wedge d\theta$

$R^2 \xrightarrow{\varphi} R^2 \quad (x, y) \to (r, \theta)$

$d x \wedge \ d \quad d A = \ r\ d\wedge r\ \ \theta$

$\varphi^* dA = dx \wedge dy$

§4 Spherical coordinates

$x = r\sin\theta\cos\varphi, y = r\sin\theta\sin\varphi, z = r\cos\theta$

Then $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The volume form $dV = \sqrt{\det g_{ij}}\, dr \wedge d\theta \wedge d\varphi = r^2 \sin\theta dr \wedge d\theta \wedge d\varphi$

§5 參考 RG1202stokes

$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$

求 $\int_{S^2} \omega =$ 其中 Ω 是實心球， $S^2 = \partial\Omega$