黎曼流形(M,g) 先給定度量，演算五個基本的式子：

1. Christoffel symbol $\Gamma^i_{jk} = \frac{1}{2} g^{il} (\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})$
2. Covariant derivative ∇_μ

   共變微分 (covariant derivative of Y along X)

   $\nabla_X Y = \sum_i (XY^i + \sum_{jk} \Gamma^i_{jk} X^j Y^k)$ ； $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma$

3. Equation of a geodesic $\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$

   定義 $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

4. Riemannian tensor $R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$
5. Einstein equation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

The curvature R of a Riemannian manifold has the following properties：

1. $R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$
   $R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2)$
2. $R(X,Y)(Z+W) = R(X,Y)Z + R(X,Y)W$
   $R(X,Y) fZ = fR(X,Y)Z$
3. R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0 Bianchi identity $R_{ijkl} + R_{iklj} + R_{iljk} = 0$
4. $R_{ijkl} = -R_{ijlk} = -R_{jikl} = R_{klij}$

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \in \chi(M)$ is the torsion of connection ∇

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z$ is the curvature of the connection ∇，measure the deviation of the map $X \to \nabla_X$ from being a Lie algebra homomorphism。

(M, g) is said to be locally conformally flat if for $\forall p \in M$，there is a local coordinate system $\{x^i\}$ in a n. b. d. U of p such that $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = v\delta_{ij}$ for some function v

Schoen-Yau

If (M, g) is a simple、connected、locally conformally flat、complete Riemannian manifold，then exists a one to one conformal map of (M, g) into the standard sphere S^n。

$\nabla_X e_j = \sum \omega_j^i(X) e_i$ ， ω_j^i are called connection forms， $\omega = [\omega_j^i]$ is called connection matrix。

$R(X,Y)e_j = \sum \Omega_j^i(X,Y)e_i$ ， Ω_j^i are called curvature forms， $\Omega = [\Omega_j^i]$ is called curvature matrix。 $\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_i^j$

If α, β are one forms，X,Y are vector fields，then

$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$

$(d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

Prove $\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_i^j$

$\nabla_X \nabla_Y e_j = \nabla_X \sum_k (\omega_j^k(Y) e_k)$ definition of connection forms

$= \sum_k X\omega_j^k(Y) e_k + \sum_k \omega_j^k \nabla_X e_k$ Leibniz rule

$= \sum_i X\omega_j^i(Y) e_i + \sum_{i\ k} \omega_j^k(Y)\omega_k^i(X$

Interchanging X and Y gives $\nabla_Y \nabla_X e_j = \sum_i Y\omega_j^i(X) e_i + \sum_{i,k} \omega_j^k(X)\omega_k^i(Y) e_i$

Furthermore $\nabla_{[X,Y]} e_j = \sum_i \omega_j^i([X,Y] e_i$

Hence, in Einstein notation,

$$\begin{aligned} R(X,Y)e_j &= \nabla_X \nabla_Y e_j - \nabla_Y \nabla_X e_j - \nabla_{[X,Y]} e_j \\ &= (X\omega_j^i(Y) - Y\omega_j^i(X) - \omega_j^i([X,Y]))e_i \\ &\quad + (\omega_k^i(X)\omega_j^k(Y) - \omega_k^i(Y)\omega_j^k(X))e_i \\ &= d\omega_j^i(X,Y)e_i + \omega_k^i \wedge \omega_j^k(X,Y)e_i \qquad \text{(by (11.3) and (11.2))} \\ &= (d\omega_j^i + \omega_k^i \wedge \omega_j^k)(X,Y)e_i. \end{aligned}$$

Comparing this with the definition of the curvature form Ω_j^i gives

$$\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^k.$$

$R_{ijkl} = \sum_m R_{ijk}^m g_{ml}$ curvature tensor

$R_{ij} = \sum_k R_{ikj}^k$ Ricci curvature tensor

$R = g^{\mu\nu} R_{\mu\nu}$ scalar curvature

Sectional curvature 截曲率

$K(\pi) := \langle R(e_1, e_2)e_2, e_1 \rangle$ where $\{e_1, e_2\}$ is an orthonormal basis of π

Prove $K(\pi) = \dfrac{\langle R(X,Y)Y, X \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}$

§ Ricci curvature and Scalar curvature

(1) 3-sphere S^3 的 Riemannian tensor，Ricci tensor，Ricci scalar
(2) $I \times S^2$ 的 Ricci tensor (RG4102-2)
(3) RG4103RicciScalarforBH

(4) Exam2009

(a) Suppose that for some smooth function ρ，we have $R_{ij} = \rho g_{ij}$ on the whole manifold M。Show that ρ is constant and $\rho = \dfrac{R}{n}$，n>2

(b) Let $(R^2, g(t))$ be a complete Riemannian surface with $g(t) = \dfrac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$

Show that

(1) In polar coordinates (r, θ)，we may rewrite

$$g(0) = ds^2 + \tanh^2 s d\theta^2, s = \log(r + \sqrt{1 + r^2})$$

(2) The scalar curvature of $(R^2, g(0))$，$R_0 = \dfrac{4}{1 + r^2}$

(3) Find 1-parameter group of conformal diffeomorphisms $\varphi_t : R^2 \to R^2$

Such that $g(t) = \varphi_t^* g(0)$

(5) Exam2015 $\mathrm{I}\times\mathrm{S}^2$

Consider the metric $g = A^2(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$ on $M = I\times S^2$

Where r is a local coordinates on $I\subset R$ and (θ,ϕ) are spherical local coordinates on S^2

(a) Compute the Ricci curvature and the scalar curvature of this metric

(b) What happens when $A(r)=\dfrac{1}{\sqrt{1-r^2}}$ ？

(c) What happens when $A(r)=\dfrac{1}{\sqrt{1+r^2}}$ ？

(d) For which fnction A® is the scalar curvature constant？

(6) Exam2018

1.

(a) Suppose that (M,g) is a 3-dimensional Riemannian manifold which is Ricci flat。
Does it have to be flat？
If your answer is yes，give a proof。If not，give a counter-example。
What if M is 4-dimensional Ricci flat manifold？

(b) Suppose that (M,g) is a 3-dimensional Riemannian manifold which is Ricci flat。
Does it have to be flat？
If your answer is yes，give a proof。If not，give a counter-example。
What if M is 4-dimentional Ricci flat manifold？Explain the reason breifly。

2. Let c be a non-negative conatant。Consider the following metric on R^n，

$$g_c = \frac{\sum_{i=1}^{n}(dx^i)^2}{(1+\frac{c}{4}\sum_{i=1}^{n}(x^i)^2)^2}$$

(a) Calculate the sectional curvatures of (R^n, g_c)

For which values of c is (R^n, g_c) complete？Justify your answer。

(7) Exam2019 這裡有些參考資料

1. On R^3，consider the following metric

$$ds^2 = dx^2 + dy^2 + (dz + \sin z dx + \cos z dy)^2$$

(a) Calculate the Riemann curvatre tensor of ds^2
(b) Denote the 1-form $dz+\sin zdx+\cos zdy$ by α 。
Can you find a regular surface Σ passing through the origin，and $T_p\Sigma\subset\ker(\alpha|_p)$ for every $p\in\Sigma$? Justify your answer。
(c) Same question as (b) for the 1-form $\beta=dz+zdx$
Can you find a regular surface Σ passing the origin，and $T_p\Sigma\subset\ker(\beta|_p)$ for every $p\in\Sigma$? Justify your answer。

1. https://profoundphysics.com/the-ricci-tensor/ 這裡有詳細的解說與例子：
The Ricci tensor represents how a volume in a curved space differs from a volume in Euclidean space。
In particular， the Ricci tensor measures how a volume between geodesics changes due to curvature。
In general relativity，the Ricci tensor represents volume changes due to gravitational tides。

Examples
1. 2-sphere S^2

$$ds^2=r^2d\theta^2+r^2\sin^2\theta d\phi^2$$

$$g_{ij}=\begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}，\Gamma^1_{ij}=\begin{pmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{pmatrix}，\Gamma^2_{ij}=\begin{pmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{pmatrix}$$

the Ricci tensor $R_{ij}=\dfrac{g_{ij}}{r^2}=\begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

2. Ricci tensor for the Schwarzschild metric
The Schwarzschild metric is a solution of Einstein's field equations in a vacuum。

In particular, it described the **spacetime around a spherically symmetric mass**。

Now， since it is a vacuum solution， the energy-momentum tensor on the right-hand side of Einstein's equations is zero。

$$R_{\mu\nu}=0$$

3. Ricci tensor for the Kerr metric
4. Ricci tensor for the Roberson-Walker (FRW) metric

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)\,r^2 d\theta^2 + a^2(t)\,r^2 \sin^2\theta d\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2r^2 & 0 \\ 0 & 0 & 0 & a^2r^2\sin^2\theta \end{pmatrix}$$

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{a\dot{a}}{c(1-kr^2)} & 0 & 0 \\ 0 & 0 & \frac{1}{c}a\dot{a}r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{c}a\dot{a}r^2\sin^2\theta \end{pmatrix}$$

$\dot{a} = \dfrac{da}{dt}$, and c is the speed of light

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} 0 & \frac{\dot{a}}{ca} & 0 & 0 \\ \frac{\dot{a}}{ca} & \frac{kr}{1-kr^2} & 0 & 0 \\ 0 & 0 & -r\left(1-kr^2\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-kr^2\right) \end{pmatrix}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{\dot{a}}{ca} & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \frac{\dot{a}}{ca} & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{\dot{a}}{ca} \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \frac{\dot{a}}{ca} & \frac{1}{r} & \cot\theta & 0 \end{pmatrix}$$

Time component $R_{00} = -\dfrac{3}{c^2}\dfrac{\ddot{a}}{a}$, space component $R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2kc^2)\dfrac{g_{ij}}{a^2}$

$$R_{\mu\nu} = \begin{pmatrix} -\frac{3}{c^2}\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{a\ddot{a}+2\dot{a}^2+kc^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & \left(a\ddot{a}+2\dot{a}^2+kc^2\right)r^2 & 0 \\ 0 & 0 & 0 & \left(a\ddot{a}+2\dot{a}^2+kc^2\right)r^2\sin^2\theta \end{pmatrix}$$

5.

Ricci tensor for the Reissner-Nordstrom metric (a black hole)

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4} \quad \& \quad r_s = \frac{2GM}{c^2}$$

以下 略　　　by Ville Hirvonen