

### § 3- sphere

1. 穩定的封閉 CMC 必為球面 J. L. Barbosa & M. do Carmo 1984
2. 宇宙可能 99%是  $S^3$
3. Poincare 猜想 任何一個單連通 閉(closed) 3 維流形一定跟  $S^3$  拓撲等價

$$4. R_{ij} = (n-1)g_{ij} \text{ where } n=3, g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi & 0 \\ 0 & 0 & \sin^2 \psi \sin^2 \theta \end{pmatrix}$$

$$S^3, x^\mu = (\psi, \theta, \phi), ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

求

$$(a) \Gamma_{jk}^i = \frac{1}{2} g^{il} \left\{ \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right\}$$

(b) 求 Riemannian tensor , Ricci tensor , Ricci scalar

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\lambda}^\rho \Gamma_{\nu\sigma}^\lambda - \Gamma_{\nu\lambda}^\rho \Gamma_{\mu\sigma}^\lambda$$

$$R_{ijk}^l = \Omega_i^l(E_j, E_k)$$

(c) Show that  $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$  is obeyed by this metric , confirming that the 3-sphere is a maximally symmetric space .

$$E_1 = \frac{\partial}{\partial \psi}, E_2 = \frac{1}{\sin \psi} \frac{\partial}{\partial \theta}, E_3 = \frac{1}{\sin \psi \sin \theta} \frac{\partial}{\partial \phi}$$

$$\omega^1 = d\psi, \omega^2 = \sin \psi d\theta, \omega^3 = \sin \psi \sin \theta d\phi$$

$$\text{Then } g = \sum_i \omega^i \otimes \omega^i$$

Cartan formula :

$$d\omega^i = \sum_j \omega^j \wedge \omega_j^i, \omega_i^j + \omega_j^i = 0, \Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

$$d\omega^1 = 0, d\omega^2 = \cos \psi d\psi \wedge d\theta, d\omega^3 = \cos \psi \sin \theta d\psi \wedge d\phi + \sin \psi \cos \theta d\theta \wedge d\phi$$

$$d\omega^1 = \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1 = 0 \Rightarrow d\theta \wedge \omega_1^2 + \sin \theta d\phi \wedge \omega_1^3 = 0$$

$$d\omega^2 = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 = d\psi \wedge \omega_1^2 + \sin\psi \sin\theta d\phi \wedge \omega_3^2 = \cos\psi d\psi \wedge d\theta$$

$$\therefore \omega_1^2 = \cos\psi d\theta$$

$$d\omega^3 = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \cos\psi \sin\theta d\psi \wedge d\phi + \sin\psi \cos\theta d\theta \wedge d\phi$$

$$= d\psi \wedge \omega_1^3 + \sin\psi d\theta \wedge \omega_2^3$$

$$\therefore \omega_1^3 = \cos\psi \sin\theta d\phi, \quad \omega_2^3 = \cos\theta d\phi$$

$$\Omega_j^i = d\omega_j^i + \omega_k^i \wedge \omega_j^k$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = -\omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = -\omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = -\omega^2 \wedge \omega^3$$

$$R_{ijk}^l = \Omega_i^l(E_j, E_k), \quad R_{iji}^j = \Omega_i^j(E_j, E_i)$$

$$R_{212}^1 = \sin^2\psi, R_{221}^1 = -\sin^2\psi, \quad R_{313}^1 = \sin^2\psi \sin^2\theta, R_{331}^1 = -\sin^2\psi \sin^2\theta$$

$$R_{121}^2 = \Omega_1^2(E_1, E_2) = 1, \quad R_{112}^2 = -1, \quad R_{323}^2 = \sin^2\psi \sin^2\theta, R_{332}^2 = -\sin^2\psi \sin^2\theta$$

$$R_{131}^3 = \Omega_1^3(E_1, E_3) = 1, \quad R_{113}^3 = -1, \quad R_{232}^3 = \Omega_2^3(E_2, E_3) = \sin^2\psi, \quad R_{223}^3 = -\sin^2\psi$$

$$R_{\theta\psi\theta}^\psi = \frac{\partial\Gamma_{\theta\theta}^\psi}{\partial\psi} - \frac{\partial\Gamma_{\theta\psi}^\psi}{\partial\theta} + \Gamma_{\lambda\psi}^\psi \Gamma_{\theta\theta}^\lambda - \Gamma_{\lambda\theta}^\psi \Gamma_{\theta\psi}^\lambda, \quad \text{where } \lambda = \psi, \theta, \phi$$

$$= \sin^2\psi \quad \text{has been confirmed } \circ$$

$$\text{Then } R_{212}^1 = \Omega_1^2(E_1, E_2) = (\omega^1 \wedge \omega^2)(E_1, E_2) = ? \dots$$

Cartan formula 的方法不知道哪裡出了錯!

正確的結果應該是

$$R_{ij} = (n-1)g_{ij} \quad \text{where } n=3, \quad g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin^2 \psi & 0 \\ 0 & 0 & \sin^2 \psi \sin^2 \theta \end{pmatrix}$$

The variational principle provides a convenient way to actually calculate the Christoffel symbols for a given metric °

$$I = \frac{1}{2} \int f d\tau = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

$$ds^2 = d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\phi^2)$$

$$I = \frac{1}{2} \int \left[ \left( \frac{d\psi}{d\tau} \right)^2 + \sin^2 \psi \left( \frac{d\theta}{d\tau} \right)^2 + \sin^2 \psi \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2 \right] d\tau$$

$S(q) = \int L(t, q(t), \dot{q}(t)) dt$  , If it independent of t , then the E-L equation are

$$\frac{\partial L}{\partial q^i} - \frac{d}{d\tau} \frac{\partial L}{\partial \dot{q}^i} = 0, \quad \text{where } L = (\dot{\psi})^2 + \sin^2 \psi (\dot{\theta})^2 + \sin^2 \psi \sin^2 \theta (\dot{\phi})^2$$

For  $\psi$  , the E-L equation is  $\frac{\partial L}{\partial \psi} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\psi}} \right) = 0$

$$\frac{\partial L}{\partial \psi} = 2 \sin \psi \cos \psi (\dot{\theta})^2 + 2 \sin \psi \cos \psi \sin^2 \theta (\dot{\phi})^2, \quad \text{and} \quad \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\psi}} \right) = \frac{d}{d\tau} (2\dot{\psi}) = 2\ddot{\psi}$$

$$2 \sin \psi \cos \psi (\dot{\theta})^2 + 2 \sin \psi \cos \psi \sin^2 \theta (\dot{\phi})^2 - 2\ddot{\psi} = 0, \quad ,$$

$$\ddot{\psi} - \sin \psi \cos \psi (\dot{\theta})^2 - \sin \psi \cos \psi \sin^2 \theta (\dot{\phi})^2 = 0$$

The geodesic equation are  $\ddot{x}^k + \Gamma_{ij}^k \dot{x}^i \dot{x}^j = 0$  , here  $x^k = \psi$

The geodesic equation is  $\ddot{\psi} + \Gamma_{\theta\theta}^\psi (\dot{\theta})^2 + \Gamma_{\phi\phi}^\psi (\dot{\phi})^2 = 0$

Thus we have  $\Gamma_{\theta\theta}^\psi = -\sin \psi \cos \psi, \Gamma_{\phi\phi}^\psi = -\sin \psi \cos \psi \sin^2 \theta$

For  $\theta$ , the Euler equation is  $\frac{\partial L}{\partial \theta} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = 0$

$$\frac{\partial L}{\partial \theta} = 2 \sin^2 \psi \sin \theta \cos \theta (\dot{\phi})^2$$

$$\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{d}{d\tau} (\sin^2 \psi (2\dot{\theta})) = 2 \sin \psi \cos \psi \dot{\psi} (2\dot{\theta}) + 2 \sin^2 \psi \ddot{\theta}$$

$$\ddot{\theta} + 2 \cot \psi \dot{\theta} \dot{\psi} - \sin \theta \cos \theta (\dot{\phi})^2 = 0$$

Compare with  $\ddot{\theta} + \Gamma_{ij}^{\theta} \dot{x}^i \dot{x}^j = 0$ ,

We have  $\Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi$ ,  $\Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$

Again for  $\phi$ , the Euler equation is  $\frac{\partial L}{\partial \phi} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 0$ , we have

$$\ddot{\phi} + 2 \cot \psi \dot{\psi} \dot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0$$

So  $\Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi$ ,  $\Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$

We have  $\Gamma_{\theta\theta}^{\psi} = -\sin \psi \cos \psi$ ,  $\Gamma_{\phi\phi}^{\psi} = -\sin \psi \cos \psi \sin^2 \theta$

$$\Gamma_{\psi\theta}^{\theta} = \Gamma_{\theta\psi}^{\theta} = \cot \psi, \quad \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$$

$$\Gamma_{\psi\phi}^{\phi} = \Gamma_{\phi\psi}^{\phi} = \cot \psi, \quad \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

(b) The Riemann tensor components are

$$R_{\sigma\mu\nu}^{\rho} = \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda}$$

$$R_{\theta\psi\theta}^{\psi} = \sin^2 \psi, \quad R_{\theta\theta\psi}^{\psi} = -\sin^2 \psi, \quad R_{\phi\psi\phi}^{\psi} = \sin^2 \psi \sin^2 \theta, \quad R_{\phi\phi\psi}^{\psi} = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\psi\theta}^{\theta} = -1, \quad R_{\psi\theta\psi}^{\theta} = 1, \quad R_{\phi\theta\phi}^{\theta} = \sin^2 \psi \sin^2 \theta, \quad R_{\phi\phi\theta}^{\theta} = -\sin^2 \psi \sin^2 \theta$$

$$R_{\psi\psi\phi}^{\phi} = -1, \quad R_{\psi\phi\psi}^{\phi} = 1, \quad R_{\theta\theta\phi}^{\phi} = -\sin^2 \psi, \quad R_{\theta\phi\theta}^{\phi} = \sin^2 \psi$$

例如

$$\begin{aligned}
R_{\psi\psi}^{\psi} &= \partial_{\psi} \Gamma_{\phi\phi}^{\psi} - \partial_{\phi} \Gamma_{\psi\phi}^{\psi} + \Gamma_{\psi\lambda}^{\psi} \Gamma_{\phi\phi}^{\lambda} - \Gamma_{\phi\lambda}^{\psi} \Gamma_{\psi\phi}^{\lambda} \\
&= \partial_{\psi} (-\sin \psi \cos \psi \sin^2 \theta) - \partial_{\phi} 0 + 0 - (-\sin \psi \cos \psi \sin^2 \theta)(\cot \psi) \\
&= -\cos^2 \psi \sin^2 \theta + \sin^2 \psi \cot^2 \theta = \sin^2 \psi
\end{aligned}$$

$$\begin{aligned}
R_{\psi\psi}^{\theta} &= \partial_{\theta} \Gamma_{\psi\psi}^{\theta} - \partial_{\psi} \Gamma_{\psi\theta}^{\theta} + \Gamma_{\theta\lambda}^{\theta} \Gamma_{\psi\psi}^{\lambda} - \Gamma_{\psi\lambda}^{\theta} \Gamma_{\psi\theta}^{\lambda} \\
&= -\partial_{\theta} (\cot \psi) - (\psi \cot \psi) = \csc^2 \psi - \cot^2 \psi = 1
\end{aligned}$$

$$\begin{aligned}
R_{\theta\theta}^{\phi} &= \partial_{\theta} \Gamma_{\theta\phi}^{\phi} - \partial_{\phi} \Gamma_{\theta\theta}^{\phi} + \Gamma_{\theta\lambda}^{\phi} \Gamma_{\theta\phi}^{\lambda} - \Gamma_{\phi\lambda}^{\phi} \Gamma_{\theta\theta}^{\lambda} \\
&= -\cos^2 \theta + \cos^2 \theta - \sin^2 \theta - (\sin \psi \cos \psi \sin^2 \theta)^2 = -\sin^2 \theta
\end{aligned}$$

$$R_{\psi\psi}^{\lambda} = R_{\psi\lambda\psi}^{\lambda} = 1 + 1 = 2, \quad R_{\theta\theta}^{\lambda} = R_{\theta\lambda\theta}^{\lambda} = \sin^2 \psi + \sin^2 \psi = 2 \sin^2 \psi$$

$$R_{\phi\phi}^{\lambda} = R_{\phi\lambda\phi}^{\lambda} = \sin^2 \psi \sin^2 \theta + \sin^2 \psi \sin^2 \theta = 2 \sin^2 \psi \sin^2 \theta$$

The Ricci tensor is twice the metric  $R_{\mu\nu} = 2g_{\mu\nu}$

The Ricci scalar  $R = g^{\mu\mu} R_{\mu\mu} = 6$

(c) Show that  $R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)} (g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu})$  is obeyed by this metric, confirming that the 3-sphere is a maximally symmetric space.

[Foundation of Differential Geometry VII] by Kobayashi Nomizu

A Riemannian manifold is called Einstein if  $S = \rho g$ , where  $S$  is the Ricci tensor and  $\rho$  is a constant.

p.35 Let  $M$  be a hypersurface immersed in  $R^{n+1}$ , at each point of  $M$ , the Ricci tensor  $S$  is given by  $S(X, Y) = g(AX, Y) \text{trace} A - g(A^2 X, Y)$ ,  $X, Y \in T_x(M)$

For  $n \geq 3$ , if  $M$  is Einstein then  $\rho \geq 0$

(1)  $\rho = 0 \dots$

(2)  $\rho > 0$   $M$  is locally a hypersphere

參考

1. [Spacetime and Geometry] Ch3 EX08 and EX16

2. [RG4102]---  $I \times S^2$   $g = A(r)^2 dt^2 + r^2 d\Omega^2$

$S^2$  做為一個 Riemannian manifold  $g = d\theta^2 + \sin^2 \theta d\phi^2$  induced from  $R^3$

3. [Wormhole metric](#) by Ellis

$c^2 dt^2 = d\rho^2 + (\rho^2 + n^2)(d\theta^2 + \sin^2 \theta d\phi^2)$  where  $n$  is the drainhole parameter

4.  $ds^2 = -c^2 dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2)$  by MT wormhole

5. [Everything Wormhole](#)

6. [Ricci curvature](#)

7. Geometry of 3-Sphere [Garret Sobczyk](#) [What's a Pauli matrix]  
[Geometric Algebra---Spinors]