

§  $S^2$  &  $I \times S^2$

[An Introduction to Riemannian Geometry ] Ex 3.3 p.106 (4)

- (4) Consider the usual local coordinates  $(\theta, \varphi)$  in  $S^2 \subset \mathbb{R}^3$  defined by the parameterization  $\phi : (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3$  given by

$$\phi(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).$$

- Using these coordinates, determine the expression of the Riemannian metric induced on  $S^2$  by the Euclidean metric of  $\mathbb{R}^3$ .
- Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.
- Show that the equator is the image of a geodesic.
- Show that any rotation about an axis through the origin in  $\mathbb{R}^3$  induces an isometry of  $S^2$ .
- Show that the images of geodesics of  $S^2$  are great circles.
- Find a **geodesic triangle** (i.e. a triangle whose sides are images of geodesics) whose internal angles add up to  $\frac{3\pi}{2}$ .
- Let  $c : \mathbb{R} \rightarrow S^2$  be given by  $c(t) = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0)$ , where  $\theta_0 \in (0, \frac{\pi}{2})$  (therefore  $c$  is not a geodesic). Let  $V$  be a vector field parallel along  $c$  such that  $V(0) = \frac{\partial}{\partial \theta}$  ( $\frac{\partial}{\partial \theta}$  is well defined at  $(\sin \theta_0, 0, \cos \theta_0)$  by continuity). Compute the angle by which  $V$  is rotated when it returns to the initial point. (**Remark:** The angle you have computed is exactly the angle by which the oscillation plane of the **Foucault pendulum** rotates during a day in a place at latitude  $\frac{\pi}{2} - \theta_0$ , as it tries to remain fixed with respect to the stars on a rotating Earth).
- Use this result to prove that no open set  $U \subset S^2$  is isometric to an open set  $W \subset \mathbb{R}^2$  with the Euclidean metric.
- Given a geodesic  $c : \mathbb{R} \rightarrow \mathbb{R}^2$  of  $\mathbb{R}^2$  with the Euclidean metric and a point  $p \notin c(\mathbb{R})$ , there exists a unique geodesic  $\tilde{c} : \mathbb{R} \rightarrow \mathbb{R}^2$  (up to reparameterization) such that  $p \in \tilde{c}(\mathbb{R})$  and  $c(\mathbb{R}) \cap \tilde{c}(\mathbb{R}) = \emptyset$  (**parallel postulate**). Is this true in  $S^2$ ?

$\varphi : M \rightarrow N$  is an immersion  $\circ (N, g)$  is a Riemannian manifold , then  $\varphi^* g$  is a Riemannian metric in M induced by  $\varphi$

(a)  $\phi : S^2 \rightarrow \mathbb{R}^3$  ,  $\phi : (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3$

Then  $\phi^* g = d\theta^2 + \sin^2 \theta d\varphi^2$  is the induced metric on  $S^2$

$$(\mathbb{R}^3, g) \quad g = (dx)^2 + (dy)^2 + (dz)^2$$

$$(S^2, \tilde{g}) \quad \tilde{g} = \phi^* g$$

$$\frac{\partial}{\partial \theta} = \frac{\partial \phi}{\partial \theta} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial \phi}{\partial \varphi} = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0)$$

$$\tilde{g}_{11} = g_{\theta\theta} = \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right\rangle = 1,$$

$$\tilde{g}_{12} = \tilde{g}_{21} = g_{\theta\varphi} = g_{\varphi\theta} = \left\langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \right\rangle = 0, \quad \tilde{g}_{22} = g_{\varphi\varphi} = \left\langle \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \right\rangle = \sin^2 \theta$$

$$\therefore \phi^* g = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi$$

Or, on  $S^2$   $x = \sin \theta \cos \varphi$ ,  $y = \sin \theta \sin \varphi$ ,  $z = \cos \theta$

$$dx = \cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi$$

$$dy = \cos \theta \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi$$

$$dz = -\sin \theta d\theta$$

Then  $dx^2 + dy^2 + dz^2 = \dots = d\theta^2 + \sin^2 \theta d\varphi^2$

$$(S^2, h) \xrightarrow{\phi} (R^3, g), \quad g = dx^2 + dy^2 + dz^2, \quad h = d\theta^2 + \sin^2 \theta d\varphi^2$$

$$\phi^* g = d\theta^2 + \sin^2 \theta d\varphi^2 = h$$

[RG3103InducedMetric]

(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2 \theta} \end{pmatrix}$$

$$\Gamma_{\varphi\varphi}^{\theta} = \frac{1}{2} \sum_{l=1}^2 g^{\theta l} \left( \frac{\partial g_{\varphi l}}{\partial \varphi} + \frac{\partial g_{\varphi l}}{\partial \varphi} - \frac{\partial g_{\varphi\varphi}}{\partial x^l} \right) = \dots = \frac{1}{2} \left( -\frac{\partial \sin^2 \theta}{\partial \theta} \right) = -\sin \theta \cos \theta$$

$$\text{同理 } \Gamma_{\theta\varphi}^{\varphi} = \Gamma_{\varphi\theta}^{\varphi} = \cot \theta$$

Or let  $L = (\dot{\theta})^2 + \sin^2 \theta (\dot{\varphi})^2$ , consider the Euler equation

$$\frac{\partial L}{\partial \theta} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = 0 \Rightarrow 2 \sin \theta \cos \theta (\dot{\varphi})^2 - \frac{d}{d\tau} (2\dot{\theta}) = 0$$

$$\ddot{\theta} - \sin \theta \cos \theta (\dot{\phi})^2 = 0 \quad \text{imply} \quad \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$$

$$\frac{\partial L}{\partial \phi} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = 0 \Rightarrow \frac{d}{d\tau} (\sin^2 \theta \times (2\dot{\phi})) = 0$$

$$2 \sin \theta \cos \theta \dot{\theta} \dot{\phi} + \sin^2 \theta \ddot{\phi} = 0 \quad \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0 \quad \text{imply} \quad \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

(c) Show that the equator is the image of a geodesic

$$\text{Geodesic equation} \quad \ddot{x}^i + \sum_{j,k} \Gamma_{jk}^i \dot{x}^j \cdot \dot{x}^k = 0 \quad \text{for } i=1, 2$$

$$\text{得} \quad \ddot{\theta} - \sin \theta \cos \theta (\dot{\phi})^2 = 0, \quad \ddot{\phi} + 2 \cot \theta \dot{\theta} \dot{\phi} = 0$$

$$\text{赤道} \begin{cases} \theta(t) = \frac{\pi}{2} \\ \phi(t) = t \end{cases}$$

(d) Show that any rotation about an axis through the origin in  $\mathbb{R}^3$  induces an isometry of  $S^2$

Any rotation about an axis through the origin in  $\mathbb{R}^3$  is an isometry of  $\mathbb{R}^3$  which preserves  $S^2$ . Since we are considering the metric in  $S^2$  induced by the Euclidean metric on  $\mathbb{R}^3$ , it is clear that such a rotation will determine an isometry of  $S^2$ .

(e) Show that the images of geodesics of  $S^2$  are great circles

Given a point  $p \in S^2$  and a vector  $v \in T_p S^2$ , there exists a rotation  $R : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $R(p) = (1, 0, 0)$  and  $R(v) = (0, 1, 0)$ . The geodesic with these initial conditions is clearly the curve  $c$  given in coordinates by  $\hat{c}(t) = (\theta(t), \varphi(t)) = (\frac{\pi}{2}, t)$ , whose image is the equator. By Exercise 3.3(3), the geodesic with initial condition  $v \in T_p S^2$  must be  $R^{-1} \circ c$ . Since the image of  $c$  is the intersection of  $S^2$  with the plane  $z = 0$ , the image of  $R^{-1} \circ c$  is the intersection of  $S^2$  with some plane through the origin, i.e. a great circle.

(f) Find a geodesic triangle whose internal angles add up to  $\frac{3\pi}{2}$

For example the triangle with vertices  $(1,0,0)$ ,  $(0,1,0)$ ,  $(0,0,1)$

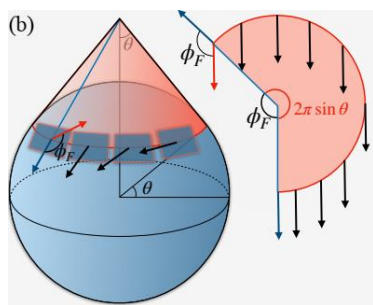
(g) Let  $c : \mathbb{R} \rightarrow S^2$  be given by  $c(t) = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0)$ , where

$\theta_0 \in (0, \frac{\pi}{2})$  (therefore  $c$  is not a geodesic) ◦ Let  $V$  be a vector field parallel along  $c$

such that  $V(0) = \frac{\partial}{\partial \theta}$  ( $\frac{\partial}{\partial \theta}$  is well defined at  $(\sin \theta_0, \cos \theta_0)$  by continuity) ◦

Compute the angle by which  $V$  is rotated when it returns to the initial point ◦

[RG3304Foucault]



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[From the geometry of Foucault pendulum to the topology of planetary waves]

The metric on  $S^2$  is  $ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$  and

$$\Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta, \quad \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

一個 vector field  $V^{\mu}$  沿曲線  $x^{\mu}(\lambda)$  平行移動則  $\frac{d}{d\lambda} V^{\mu} + \Gamma_{ij}^{\mu} \frac{dx^i}{d\lambda} V^j = 0$

Along the curve  $c$ ,  $\theta = \theta_0$  is a constant,  $\phi = t$  then  $\dot{\theta} = 0, \dot{\phi} = \frac{d\phi}{dt} = 1$

The equations for paralleltransport are  $\dot{V}^i + \sum_{j,k} \Gamma_{jk}^i x^j V^k = 0$

$$\dot{V}^{\theta} + \sum_{ij} \Gamma_{ij}^{\theta} x^i V^j = 0, \quad \dot{V}^{\phi} + \Gamma_{\phi\phi}^{\theta} \dot{\phi} V^{\phi} = 0, \quad \dot{V}^{\theta} - \sin \theta_0 \cos \theta_0 V^{\phi} = 0 \dots (1)$$

$$\dot{V}^{\phi} + \sum_{i,j} \Gamma_{ij}^{\phi} x^i V^j = 0, \quad \dot{V}^{\phi} + \Gamma_{\theta\theta}^{\phi} \dot{\theta} V^{\phi} + \Gamma_{\phi\theta}^{\phi} \dot{\phi} V^{\theta} = 0, \quad \dot{V}^{\phi} + \cot \theta_0 V^{\theta} = 0 \dots (2)$$

Since  $\dot{\theta} = 0, \dot{\phi} = 1$

$$(1) \text{式兩邊對 } t \text{ 微分, } \ddot{V}^{\theta} = \sin \theta_0 \cos \theta_0 \dot{V}^{\phi} = -\sin \theta_0 \cos \theta_0 \cot \theta_0 V^{\theta} = -\cos^2 \theta_0 V^{\theta}$$

解此微分方程,  $V^{\theta} = A \cos(t \cos \theta_0) + B \sin(t \cos \theta_0)$

$$V^{\phi} = \frac{1}{\sin \theta_0 \cos \theta_0} \dot{V}^{\theta} = \frac{1}{\sin \theta_0 \cos \theta_0} (-\cos \theta_0 A \sin(t \cos \theta_0) + \cos \theta_0 B \cos(t \cos \theta_0))$$

$$V(0) = \frac{\partial}{\partial \theta} = (V^\theta(0), V^\phi(0)) = (1, 0) \Rightarrow V^\theta(0) = 1, V^\phi(0) = 0$$

Imply  $A=1, B=0$ , so  $V^\theta = \cos(t \cos \theta_0), V^\phi = -\frac{1}{\sin \theta_0} \sin(t \cos \theta_0)$

Note that  $|V| = g_{\mu\nu} V^\mu V^\nu = V^\theta V^\theta + \sin^2 \theta_0 V^\phi V^\phi = 1$

假設  $V(0), V(2\pi)$  的夾角為  $\alpha$ ,

$$V(0) = \frac{\partial}{\partial \theta}, V(2\pi) = \cos(2\pi(\cos \theta_0)) \frac{\partial}{\partial \theta} - \frac{1}{\sin \theta_0} \sin(2\pi(\cos \theta_0)) \frac{\partial}{\partial \phi}$$

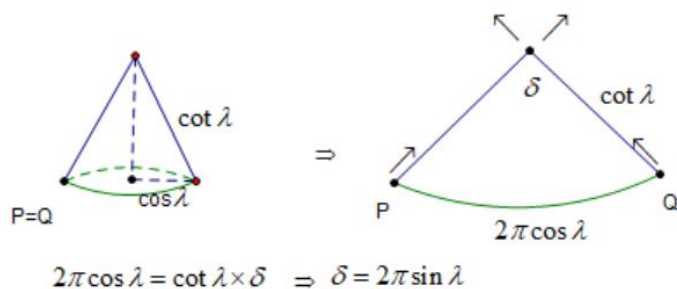
則  $\cos \alpha = \langle V(0), V(2\pi) \rangle = \cos(2\pi \cos \theta_0)$

所以  $\alpha = 2\pi \cos \theta_0$  或  $2\pi(1 - \cos \theta_0)$

若傅科擺位於北緯  $\lambda$  度 ( $\lambda = \frac{\pi}{2} - \theta_0$ ), 則擺動方向每 24 小時轉動  $2\pi \sin \lambda$

參考 [大域微分幾何] p.156~162

Do Carmo 第二章習作 p.58



(h) Use this result to prove that no open set  $U \subset S^2$  is isometric to an open set  $W \subset \mathbb{R}^2$  with the Euclidean metric

Using the fact that any point on  $S^2$  can be carried to  $(0, 0, 1)$  by an appropriate isometry, we just have to show that no open neighborhood  $U \subset S^2$  of  $(1, 0, 0)$  is isometric to an open set  $V \subset \mathbb{R}^2$  with the Euclidean metric. Now any such neighborhood contains the image of a curve  $c(t)$  as given in (g)

(for  $\theta_0 > 0$  sufficiently small). If  $U$  were isometric to  $W$ , the Levi-Civita connection on  $U$  would be the trivial connection, and hence the parallel vector field  $V(t)$  in (g) would satisfy  $V(0) = V(2\pi)$ . Since this is not true for any  $\theta_0 \in (0, \frac{\pi}{2})$ ,  $U$  cannot be isometric to  $W$ .

(i) 求  $S^2$  的 Gauss curvature

$$ds^2 = a^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

$$\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta, \quad \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \cot \theta$$

$$\begin{aligned} R_{\phi\theta\phi}^\theta &= \partial_\theta \Gamma_{\phi\phi}^\theta - \partial_\phi \Gamma_{\theta\phi}^\theta + \Gamma_{\theta\lambda}^\theta \Gamma_{\phi\phi}^\lambda - \Gamma_{\phi\lambda}^\theta \Gamma_{\theta\phi}^\lambda \\ &= (\sin^2 \theta - \cos^2 \theta) + 0 - 0 - (\theta \sin \theta \cos \theta) = 2 \sin \theta \cos \theta \end{aligned}$$

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda} R_{\phi\theta\phi}^\lambda = g_{\theta\theta} R_{\phi\theta\phi}^\theta = a^2 \sin^2 \theta$$

All the components of the Riemann tensor either vanish or are related to this one by symmetry.

The Ricci tensor  $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$

$$R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1$$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2 \theta$$

The Ricci scalar

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}$$

p.139 Ex2.8 (6)

(6) Consider the metric

$$g = A^2(r) dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2 \theta d\varphi \otimes d\varphi$$

on  $M = I \times S^2$ , where  $r$  is a local coordinate on  $I \subset \mathbb{R}$  and  $(\theta, \varphi)$  are spherical local coordinates on  $S^2$ .

- Compute the Ricci tensor and the scalar curvature of this metric.
- What happens when  $A(r) = (1 - r^2)^{-\frac{1}{2}}$  (that is, when  $M$  is locally isometric to  $S^3$ )?
- And when  $A(r) = (1 + r^2)^{-\frac{1}{2}}$  (that is, when  $M$  is locally isometric to the **hyperbolic 3-space**)?
- For which functions  $A(r)$  is the scalar curvature constant?

$(M, g)$  是一個黎曼流形

$$X_1 = \frac{\partial}{\partial r}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \varphi}$$

$\langle X_1, X_1 \rangle = A(r)^2, \langle X_2, X_2 \rangle = r^2, \langle X_3, X_3 \rangle = r^2 \sin^2 \theta$ ，所以取 orthonormal frames

$$\text{為 } E_1 = \frac{1}{A} \frac{\partial}{\partial r}, E_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, E_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}$$

An orthonormal coframes  $\omega^1 = A(r)dr, \omega^2 = rd\theta, \omega^3 = r \sin \theta d\varphi$

由結構方程式  $d\omega^i = \omega^j \wedge \omega_k^i$

$$d\omega^1 = 0$$

$$d\omega^2 = dr \wedge d\theta = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 = A dr \wedge \omega_1^2$$

$$\therefore \omega_1^2 = \frac{1}{A} d\theta$$

$$d\omega^3 = \sin \theta dr \wedge d\varphi + r \cos \theta d\theta \wedge d\varphi = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3$$

$$= A dr \wedge \omega_1^3 + rd\theta \wedge \omega_2^3, \therefore \omega_1^3 = \frac{\sin \theta}{A} d\varphi, \omega_2^3 = \cos \theta d\varphi$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = d\left(\frac{1}{A} d\theta\right) - \left(\frac{\sin \theta}{A} d\varphi\right) \wedge (-\cos \theta d\varphi) = \frac{-A'}{A^2} dr \wedge d\theta = \frac{-A'}{rA^3} \omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = d\left(\frac{\sin \theta}{A} d\varphi\right) - \left(\frac{1}{A} d\theta\right) \wedge (\cos \theta d\varphi)$$

$$= \left(\frac{-A' \sin \theta}{A^2} dr + \frac{\cos \theta}{A} d\theta\right) \wedge d\varphi - \frac{\theta \cos \theta}{A} d\theta \wedge d\varphi$$

$$= \frac{-A' \sin \theta}{A^2} d\theta \wedge d\varphi = \frac{-A'}{rA^3} \omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = d(\cos \theta d\varphi) - \left(-\frac{1}{A} d\theta\right) \wedge \left(\frac{\sin \theta}{A} d\varphi\right)$$

$$= (-\sin \theta d\theta \wedge d\varphi) + \frac{\sin \theta}{A^2} d\theta \wedge d\varphi$$

$$= \left(\frac{\sin \theta}{A^2} - \sin \theta\right) d\theta \wedge d\varphi = \frac{1}{r^2} \left(\frac{1}{A^2} - 1\right) \omega^2 \wedge \omega^3$$

$$R_{ij}^j = \Omega_i^j(E_i, E_j), R_{ij} = R_{kij}^k$$

$$R_{121}^2 = \Omega_1^2(E_1, E_2) = \frac{-A'}{rA^3}, \quad R_{131}^3 = \Omega_1^3(E_1, E_3) = \frac{-A'}{rA^3}, \quad R_{232}^3 = \Omega_2^3(E_2, E_3) = \frac{1}{r^2} \left( \frac{1}{A} - 1 \right)$$

$$R_{11} = R_{111}^1 + R_{211}^2 + R_{311}^3 = -R_{121}^2 - R_{131}^3 = \frac{2A'}{rA^3}$$

$$R_{22} = R_{122}^1 + R_{222}^2 + R_{322}^3 = -R_{121}^2 - R_{232}^3 = \frac{A'}{rA^3} - \frac{1}{r^2} \left( \frac{1}{A^2} - 1 \right)$$

$$R_{33} = R_{133}^1 + R_{233}^2 + R_{333}^3 = -R_{131}^3 - R_{232}^3 = \frac{A'}{rA^3} - \frac{1}{r^2} \left( \frac{1}{A^2} - 1 \right)$$

$$\text{Scalar curvature } R = R_{11} + R_{22} + R_{33} = \frac{4A'}{rA^3} - \frac{2}{r} \left( \frac{1}{A^2} - 1 \right)$$