§ The geodesic equations for a metric of a BH

We consider a charged BH surrounded quintessence with the equation of state parameter $\varepsilon = \frac{p\phi}{\sigma\phi}$, $-1 < \varepsilon < -\frac{1}{3}$, and α is the normalization constant。

Quintessence 第五元素 a hypothetical form of dark energy ，a scalar field

The metric of a charged BH reads：

$$ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Where $f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r} - \frac{\alpha}{r^{3\varepsilon+1}}$

Where M and Q represent the mass and charge of the BH respectively。

The metric represents a BH for M>Q，an extremal BH for M=Q and a naked singularity for M<Q。

The geodesic equations for the metric are given by

$$\ddot{t} + \frac{f'(r)}{f(r)}\dot{r}\dot{t} = 0 \qquad (1)$$

$$\ddot{r} + [\frac{f'(r)\dot{t}^2 + f^{-1}(r)'\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2}{2f^{-1}(r)}] = 0 \qquad (2)$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \cos\theta\sin\theta\dot{\phi}^2 = 0 \qquad (3)$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0 \qquad (4)$$

Where the prime denotes the differentiation with respect to r 。

The geodesic equations are $\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0$

Where $\Gamma^i_{jk} = \frac{1}{2}g^{il}(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})$

$$(g_{ij}) = \begin{pmatrix} f(r) & 0 & 0 & 0 \\ 0 & -\frac{1}{f(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} \frac{1}{f(r)} & 0 & 0 & 0 \\ 0 & -f(r) & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{r^2\sin^2\theta} \end{pmatrix}$$

$$X = X(t, r, \theta, \phi)$$

$$\ddot{t} + \Gamma^1_{jk} \dot{x}^j \dot{x}^k = 0$$

$$\ddot{t} + \Gamma^1_{11} \dot{t}\dot{t} + \Gamma^1_{12} \dot{t}\dot{r} + \Gamma^1_{21} \dot{r}\dot{t} + \Gamma^1_{22} \dot{r}\dot{r} + \{\ldots\}$$

$$\Gamma^1_{11} = \frac{1}{2} g^{1l} (\frac{\partial g_{1l}}{\partial x^1} + \frac{\partial g_{1l}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^l}) = 0$$ ，

$$\Gamma^1_{22} = \frac{1}{2} g^{1l} (\frac{\partial g_{2l}}{\partial x^2} + \frac{\partial g_{2l}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^l}) = \frac{1}{2} g^{11} (\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}) = 0$$

$$\Gamma^1_{12} = \frac{1}{2} g^{1l} (\frac{\partial g_{2l}}{\partial x^1} + \frac{\partial g_{1l}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^l}) = \frac{1}{2} \times \frac{1}{f(r)} \times \frac{\partial f}{\partial r} = \Gamma^1_{21}$$

所以得到(1) $\ddot{t} + \frac{f'(r)}{f(r)} \dot{t}\dot{r} = 0$

$$\ddot{r} + \Gamma^2_{jk} \dot{x}^j \dot{x}^k = 0$$

$$\Gamma^2_{11} = \frac{1}{2} g^{2l} (\frac{\partial g_{1l}}{\partial x^1} + \frac{\partial g_{1l}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^l}) = \frac{1}{2} g^{22} \times (-\frac{\partial g_{11}}{\partial x^2}) = \frac{1}{2} f(r) \frac{\partial f(r)}{\partial r}$$

$$\Gamma^2_{22} = \frac{1}{2} g^{22} (\frac{\partial g_{2l}}{\partial x^2} + \frac{\partial g_{2l}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^l}) = \frac{1}{2} g^{22} (\frac{\partial g_{22}}{\partial r} + \frac{\partial g_{22}}{\partial r} - \frac{\partial g_{22}}{\partial r}) = \frac{1}{2} f(r) \times f^{-1}(r)'$$

$$\Gamma^2_{33} = \frac{1}{2} g^{2l} (\frac{\partial g_{3l}}{\partial x^3} + \frac{\partial g_{3l}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^l}) = \frac{1}{2} (-2rf(r))$$

$$\Gamma^2_{44} = \frac{1}{2} g^{2l} (\frac{\partial g_{4l}}{\partial x^4} + \frac{\partial g_{4l}}{\partial x^4} - \frac{\partial g_{44}}{\partial x^l}) = \frac{1}{2} (-f(r))(2r \sin^2 \theta)$$

得到(2)

$$\ddot{\theta} + \Gamma^3_{jk} d\dot{x}^j \, d\dot{x}^k = 0$$

$\Gamma^3_{11} = 0 \quad \Gamma^3_{22} = 0 \quad \Gamma^3_{33} = 0 \quad \Gamma^3_{44} = -\sin\theta \cos\theta$

$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{r}$ ，其他皆為 0

得到(3) $\ddot{\theta} + \frac{2}{r} \dot{r}\dot{\theta} - \cos\theta \sin\theta \dot{\phi}^2 = 0$

$$\ddot{\phi} + \Gamma^4_{jk} d\dot{x}^j \, d\dot{x}^k = 0$$

$\Gamma^4_{24} = \Gamma^4_{42} = \frac{1}{r} \qquad \Gamma^4_{34} = \Gamma^4_{43} = \cot\theta$ ，其他皆為 0

得到(4) $\ddot{\phi} + \frac{2}{r} \dot{r}\dot{\phi} + 2\cot\theta \dot{\theta}\dot{\phi} = 0$