§ Jacobi Fields

A geodesic \( \gamma: I \to M, \gamma(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) be a geodesic \( \iff \frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} = 0 \)

\[ \text{TM} = \{ (p, v) \mid p \in M, v \in T_p M \} \text{ is a tangent bundle} \]

\( \{ U, x \} \text{ is M's coordinate system, in T} \rightarrow \text{U's coordinate get T} \rightarrow \text{M's differentiable structure} \)

\( p(x_1, \ldots, x_n), v = \sum_i y_i \frac{\partial}{\partial x^i} \), then \( (p, v) = (x_1, \ldots, x_n, y_1, \ldots, y_n) \)

A geodesic \( t \to (x_1(t), x_2(t), \frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt}) \) satisfies \( \begin{cases} \frac{dx_k}{dt} = y_k \\ \frac{dy_k}{dt} = -\Gamma^k_{ij} y_i y_j \end{cases} \)

§ Jacobi Field

A Jacobi field \( J(t) \) along geodesic \( \gamma \) satisfies the Jacobi equation for all \( t \in [0, I] \)

\( \frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0 \)

Classical writing (do Carmo)

\( J(t) \) is along \( \gamma: [0, I] \to M, t \in [0, I] \) the Jacobi field then \( J \) satisfies Jacobi equation.

\( \frac{D}{dt} \left( \frac{D}{dt} J(t) + K(t) \gamma'(t) \wedge J(t) \right) \wedge \gamma'(t) = 0 \), among \( K(t) \) is M's Gaussian curvature

Globally differential geometric writing

\( \nabla_t J + R(J, T)T = 0 \), among \( T = \frac{d\gamma}{dt} \).

\( R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z \)

Along geodesic \( C \) of Jacobi field's role lies in how quickly the geodesics near \( C \) pull away from \( C \).

(A Jacobi field along geodesic \( C \) which measures how rapidly the geodesics near \( C \) pull away from \( C \).)
§ 一個實例

\( \gamma(t) \) 是一經線 (geodesic) \( t \in [0, \pi] \)

將 \( \gamma(t) \) 做變分 \( \gamma_v(t) = \hat{\gamma}(t, v) \)

使每一條 \( \gamma_v \) 都是測地線

則 \( J = \frac{\partial \hat{\gamma}}{\partial v} \) 即 \( \gamma \) 上的一個 Jacobi 場

以下是 do Carmo 書的寫法

\( \omega(t) \) 是 \( \gamma \) 上切於緯圓的 \( C_t \) 的平行向量場且 \( |\omega| = 1 \) 且 \( \langle \omega(0), \gamma'(0) \rangle = 0 \)

則 \( J(t) = (\sin t)\omega(0) \) 是 \( \gamma \) 上的一個 Jacobi 場

\[ J(t) = (\sin t)\omega(0) \]

\[ \frac{dJ}{dt} = (\cos t)\omega(t) + (\sin t)\omega'(t), \quad |\omega| = 1, \omega \cdot \omega' = 0, \quad \text{取切部} \]

\[ \frac{DJ}{dt} = (\cos t)\omega(t), \quad \text{同理} \]

\[ \frac{D}{dt} \frac{DJ}{dt} = (-\sin t)\omega(t) \]

因為 \( \omega(t), \gamma'(t) \) 皆為平行移動 故保持度量 即 \( \langle \omega(t), \gamma'(t) \rangle = 0 \)

\[ (\gamma'(t) \wedge \omega(t)) \wedge \gamma'(t) = \omega(t) \]

所以 \( (\gamma'(t) \wedge J(t)) \wedge \gamma'(t) = J(t) \)

\[ R = K = 1 \]

\[ \frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = (\sin t)\omega(0) + (\sin t)\omega(t) = 0 \]

所以 \( J(t) = (\sin t)\omega(0) \) 是 \( \gamma \) 上的一個 Jacobi 場

注意到 \( J(\pi) = 0 \)

獲得 Jacobi 場的一個方法是測地線作移動，沿測地線 \( \gamma : [a, b] \to M \) 的任一 Jacobi 場皆可由 \( \gamma \) 通過測地線的變分得到。

[王興中 p.298]
§ 共軛點
\[ \gamma : [0, l] \to M \] 是一測地線，\( \gamma(0) = P, \ Q = \gamma(t_0), \ t_0 \in [0, l] \)
若存在一沿著 \( \gamma \) 不全為 0 的 Jacobi 場 \( J(t) \)，使得 \( J(0) = J(t_0) \)，則稱 \( Q \) 是 \( P \) 的共軛點(關於測地線 \( \gamma \) )，因此 上例中 球面上 \( Q \) 是 \( P \) 的共軛點

§ 命題與定理
命題一
\( J_1(t), J_2(t) \) 是沿著 \( \gamma : [0, l] \to M \) 的兩 Jacobi 場 則
\[ < \frac{DJ_1}{dt}, J_2(t) > - < J_1(t), \frac{DJ_2}{dt} > = const \]

命題二
\( J(t) \) 是沿著 \( \gamma : [0, l] \to M \) 的 Jacobi 場 \( J(t), \gamma'(t_1) \Rightarrow J(t), \gamma'(t_2) \) \( = 0 \) for \( t_1 \neq t_2 \)
則 \( < J(t), \gamma'(t) > = 0, \forall t \in [0, l] \)

定理
\( \gamma \) 是測地線，給定 \( J(0), \ J'(0) \), \( A, B \in T_p(M), \gamma(0) = P \)
則存在唯一一個 \( \gamma \) 上的 Jacobi 場 滿足 \( J(0) = A, \ J'(0) = B \)，其中 \( J' \) 表示 \( \nabla_\gamma J \)

辛格(John Lipton Synge 1897~1995) 是最早研究黑洞內部的物理學家之一稱 Jacobi fields 方程是測地線偏離方程 在引力波的研究中有應用。

[測地線偏離效應(geodesic deviation)]
https://kknews.cc/zh-tw/science/v94o9nl.html(盧昌海)
時空曲率的存在會導致沿相鄰測地線運動的試驗粒子之間的距離發生變化，這是所謂的測地偏離 (geodesic deviation) 效應，它是引力相互作用的一種體現。
維基百科如此說明
In general relativity, geodesic deviation describes the tendency of objects to approach or recede from one another while moving under the influence of a spatially varying gravitational field. Put another way, if two objects are set in motion along two initially parallel trajectories, the presence of a tidal gravitational force will cause the trajectories to bend towards or away from each other, producing a relative acceleration between the objects.\[1\]

Mathematically, the tidal force in general relativity is described by the Riemann curvature tensor,\[1\] and the trajectory of an object solely under the influence of gravity is called a geodesic. The geodesic deviation
Jacobi Fields

equation relates the Riemann curvature tensor to the relative acceleration of two neighboring geodesics. In differential geometry, the geodesic deviation equation is more commonly known as the Jacobi equation.

Synge 定理

若 $M$ 是一有正截曲率(sectional curvature) 緊緻的黎曼流形 則
(1) 若 $M$ 是偶數維 可定向 則 $M$ 是單連通的(simple connected)
(2) 若 $m$ 是奇數維 則 $m$ 可定向

[大域微分幾何]下卷 第 30 章 cmc 上 Jacobi 場與 Morse Index 定理

(6) Let $(M, \langle , \rangle )$ be a Riemannian manifold whose geodesics can be extended for all values of their parameters, and let $p \in M$.

(a) Let $X$ and $Y$ be the vector fields defined on a normal ball centered at $p$ as in (3.6) and (3.7). Show that $Y$ satisfies the Jacobi equation

$$
\nabla_X \nabla_X Y_i = R(X, Y_i) X,
$$

where $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, defined by

$$
R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,
$$

is called the curvature operator (cf. Chap. 4). (Remark: It can be shown that $(R(X, Y) Z)_p$ depends only on $X_p, Y_p, Z_p$).

(b) Consider a geodesic $c : \mathbb{R} \to M$ parameterized by the arclength such that $c(0) = p$. A vector field $Y$ along $c$ is called a Jacobi field if it satisfies the Jacobi equation along $c$.

$$
\frac{D^2 Y}{dt^2} = R(\dot{c}, Y) \dot{c}.
$$

Show that $Y$ is a Jacobi field with $Y(0) = 0$ if and only if

$$
Y(t) = \frac{\partial}{\partial s} \exp_p(t v(s)) \Big|_{s=0}
$$

with $v : (-\varepsilon, \varepsilon) \to T_p M$ satisfying $v(0) = \dot{c}(0)$.

(c) A point $q \in M$ is said to be conjugate to $p$ if it is a critical value of $\exp_p$. Show that $q$ is conjugate to $p$ if and only if there exists a nonzero Jacobi field $Y$ along a geodesic $c$ connecting $p = c(0)$ to $q = c(b)$ such that $Y(0) = Y(b) = 0$. Conclude that if $q$ is conjugate to $p$ then $p$ is conjugate to $q$. 
(d) The manifold \( M \) is said to have \textbf{nonpositive curvature} if \( \langle R(X, Y)X, Y \rangle \geq 0 \) for all \( X, Y \in \mathfrak{X}(M) \). Show that for such a manifold no two points are conjugate.

(e) Given a geodesic \( c : I \to M \) parameterized by the arclength such that \( c(0) = p \), let \( t_c \) be the supremum of the set of values of \( t \) such that \( c \) is the minimizing curve connecting \( p \) to \( c(t) \) (hence \( t_c > 0 \)). The \textbf{cut locus} of \( p \) is defined to be the set of all points of the form \( c(t_c) \) for \( t_c < +\infty \). Determine the cut locus of a given point \( p \in M \) when \( M \) is:

(i) the torus \( T^n \) with the flat square metric;
(ii) the sphere \( S^n \) with the standard metric;
(iii) the projective space \( \mathbb{R}P^n \) with the standard metric.

Check in these examples that any point in the cut locus is either conjugate to \( p \) or joined to \( p \) by two geodesics with the same length but different images.

(Remark: This is a general property of the cut locus—see [EM93] or [GHP04] for a proof).

[Riemann Geometry p.116 EX4.8]

§ Jacobi 場與變分法的關係

\[ \gamma(t) = \gamma_0(t) = F(0, t) \]

而假定曲線 \( \gamma \) 可連綿地畸变为

\[ \alpha(t) = \gamma_1(t) \]
\[ \beta(t) = \gamma_1(t) \]

曲線族 \( \gamma_1(t) \) 相對 \( \gamma(t) \) 的變更又可稱為 \( \gamma \) 的變分. 由 \( F(u, t) \) 的連續性, 也可固定 \( t \), 这时 \( F(u, t) = F_t(u) \) 为由 \( \alpha(t) \) 到 \( \beta(t) \) 的曲率. 下面用 \( V = \frac{\partial}{\partial t}, J = \frac{\partial}{\partial u} \) 为切于相应曲线族的切场, 假定它们间关系满足
1. Jacobi 場與 exponential map 的關係
2. Jacobi 場的物理意義
3. Jacobi 場與潮汐力(Tidal forces)

從時空曲率從理論上可以通過測地線偏離方程(geodesic deviation equation)觀測到。曲率張量表示剛體沿測地線運動時所受的潮汐力(tidal force)，其意義可由 Jacobi equation 精確描述。
1. **Geodesic deviation: the curvature tensor and Jacobi fields.**

Let $(M, g)$ be a Riemannian manifold, $p \in M$. Suppose we want to measure the “instantaneous spreading rate” of geodesic rays issuing from $p$. The natural way to do this is to consider a geodesic variation:

$$f(t, s) = \exp_p(tv(s)), \quad v(s) \in T_p M, \quad v(0) = v, \quad v'(0) = w.$$ 

Then the “spreading rate” is measured by the variation vector field $V(t)$:

$$V(t) = \frac{\partial f}{\partial s} |_{s=0} = d\exp_p(tv)[tw],$$

a vector field along the geodesic $\gamma(t) = \exp_p(tv)$. Let’s try to find a differential equation satisfied by $V$. We have:

$$\frac{D}{dt} V = \frac{D}{dt} \frac{\partial f}{\partial s} = D \frac{\partial f}{\partial s} \frac{\partial s}{\partial t} + \frac{D^2 V}{\partial t^2} = D \frac{D}{dt} \frac{\partial f}{\partial s}.$$ 

Setting $W = \partial_t f$ (a vector field along $f$), we see that $D W / \partial t \equiv 0$:

$$\frac{D}{dt} \frac{\partial f}{\partial t} = \frac{D}{dt} (d\exp_p(tv(s))[v(s)]) = \frac{D}{dt} \gamma_s(t) = 0,$$

since $\gamma_s(t) = \exp_p(tv(s))$ is a geodesic ($\gamma_s(0) = p, \gamma_s(0) = v(s)$). Thus we need to compute the vector field $X(t)$ along $f$:

$$X(t) := \frac{D}{dt} \frac{D W}{\partial s} - \frac{D}{dt} \frac{D W}{\partial t},$$

for then, along $\gamma(t)$:

$$\frac{D^2 V}{\partial t^2} = X(t).$$

We compute in a coordinate chart:

$$f(s, t) = (x^i(s, t)) \in \mathbb{R}^m, \quad W(s, t) = a_i(s, t) \partial_{x_i}.$$ 

Using the symmetry of the connection, we find:

$$X = a_i \partial_t x_k \partial_k x_j (\nabla_{\partial_k x_i} \nabla_{\partial_k x_j} \partial_{x_i} - \nabla_{\partial_k x_j} \nabla_{\partial_k x_i} \partial_{x_j}).$$

We now consider the linearity over smooth functions of this commutator of covariant derivatives. We find:

$$\nabla_{\partial_k x_i} \nabla_{\partial_j x_i} (f \partial_{x_i}) - \nabla_{\partial_j x_j} \nabla_{\partial_k x_k} (f \partial_{x_i}) = f(\nabla_{\partial_k x_i} \nabla_{\partial_j x_j} \partial_{x_i} - \nabla_{\partial_j x_j} \nabla_{\partial_k x_k} \partial_{x_i}).$$
and, assuming we have normal coordinates at \( p \) (so \( \nabla_{\partial_{x_i}} \partial_{x_i} = 0 \) at \( p \)):
\[
\nabla f \partial_{x_k} \nabla \partial_{x_j} \partial_{x_i} - \nabla \partial_{x_j} \nabla f \partial_{x_k} \partial_{x_i} = f(\nabla \partial_{x_k} \nabla \partial_{x_j} \partial_{x_i} - \nabla \partial_{x_j} \nabla \partial_{x_k} \partial_{x_i}).
\]
Thus, by linearity, we have:
\[
X(t) = \nabla_{\partial_t} \nabla \partial_{\partial_t} W - \nabla \partial_t \nabla \partial_{\partial_t} W.
\]
This suggests considering, given three vector fields \( X, Y, W \), the vector field:
\[
\nabla_X \nabla_Y W - \nabla_Y \nabla_X W.
\]
A natural question is whether this is “tensorial” (linear over smooth functions) in each of \( X, Y, W \). Starting with \( W \), we find:
\[
(\nabla_X \nabla_Y - \nabla_Y \nabla_X)(fW) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X)W + [X, Y]f.
\]
This suggests subtracting the term \( \nabla_{[X, Y]}W \). Computing again:
\[
(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})(fW) = f(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})W.
\]
This motivates the definition:

**Definition.** The \((3,1)\)-Riemann curvature tensor \( R \) assigns to three vector fields \((X, Y, Z)\) on \( M \) the vector field:
\[
R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z.
\]

**Exercise.** This assignment is linear over smooth functions in each of \( X, Y \) and \( Z \).

For a vector field \( W(t, s) \) along an immersion \( f(t, s) \), we get the **Ricci equation**:
\[
\frac{D}{Dt} \frac{D}{Ds} W - \frac{D}{Ds} \frac{D}{Dt} W = R(\partial_t f, \partial_s f)W.
\]

Returning to the vector field \( W(t, s) = \partial_t f \) along the geodesic variation \( f(t, s) \) (where the curves \( t \mapsto f(t, s) \) are geodesics), we find:
\[
\frac{D^2 V}{dt^2} = \frac{D}{Dt} \frac{D}{Dt} W = R(\partial_t f, \partial_s f)\partial_t f,
\]
and at \( s = 0 \) (since \( \partial_t f|_{s=0} = \dot{\gamma} \) and \( \partial_s f|_{s=0} = V \)):
\[
\frac{D^2 V}{dt^2} + R(V, \dot{\gamma})\ddot{\gamma} = 0.
\]
This is the **Jacobi equation** for the “geodesic deviation” vector field \( V(t) \); its solutions are **Jacobi fields** along \( \gamma(t) \).
Remark. To find the first-order initial condition for $V(t)$, consider:

$$\frac{DV}{dt}_{\mu=0} = D\frac{\partial f}{\partial s} \frac{\partial s}{\partial t}_{\mu=0, s=0} = D\frac{\partial f}{\partial s} \frac{d}{ds} \gamma_s(0) = \nu'(0) = w.$$

We conclude:

$$J(t) = d\exp_p(tv)[t\nu]$$ is the Jacobi field along $\gamma(t) = \exp_p(tv)$ with IC $J(0) = 0, \dot{J}(0) = w$.

In particular: $d\exp_p(v)[w] = J(1)$. This expresses the differential of the exponential map in terms of the solution of a differential equation along $\gamma(t)$. 
