§ Cosmology

Friedmann-Lemaitre-Robertson-Walker model

(M, g)， $g = -dt^2 + a^2(t)(\frac{1}{1-kr^2}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2)$

$$\omega^0 = dt, \omega^1 = a(t)(1-kr^2)^{-\frac{1}{2}}dr, \omega^2 = a(t)rd\theta, \omega^3 = a(t)r\sin\theta d\varphi$$

$\{\omega^0, \omega^1, \omega^2, \omega^3\}$ is an orthonormal coframe

$$d\omega^i = \omega^j \wedge \omega_j^i$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

$$R_{ij} = \sum_k R_{kij}^k \text{ ， } R_{iji}^j = \Omega_i^j(E_i, E_j)$$

$$d\omega^1 = \dot{a}(1-kr^2)^{-\frac{1}{2}}dt \wedge dr = \omega^0 \wedge \omega_0^1 + \omega^2 \wedge \omega_2^1 + \omega^3 \wedge \omega_3^1$$

$$= dt \wedge \omega_0^1 + X\theta \wedge \omega_2^3 + X\phi \wedge \text{ ， } \therefore \omega_0^1 = \dot{a}(1-kr^2)^{-\frac{1}{2}}dr = \omega_1^0$$

$$d\omega^2 = \dot{a}rdt \wedge d\theta + adr \wedge d\theta = \omega^0 \wedge \omega_0^2 + \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2$$

$$= dt \wedge \omega_0^2 + (a(1-kr^2)^{-\frac{1}{2}} dr) \wedge \omega_1^2 + X d\varphi$$

$$\therefore \omega_0^2 = \omega_2^0 = \dot{a}rd\theta \text{ ， } \omega_1^2 = (1-kr^2)^{-\frac{1}{2}}d\theta = -\omega_2^1$$

$$d\omega^3 = \dot{a}r\sin\theta dt \wedge d\varphi + a\sin\theta dr \wedge d\varphi + ar\cos\theta d\theta \wedge d\varphi$$

$$= \omega^0 \wedge \omega_0^3 + \omega^1 \wedge \omega_1^3 + \omega^2 \wedge$$

$$= dt \wedge \omega_0^3 + (a(1-kr^2)^{-\frac{1}{2}} dr) \wedge \omega_1^3 + \theta r d\varphi$$

$$\therefore \omega_0^3 = \omega_3^0 = \dot{a}r\sin\theta d\varphi \text{ ， } \omega_1^3 = -\omega_3^1 = (1-kr^2)^{\frac{1}{2}}\sin\theta d\varphi \text{ ， } \omega_2^3 = -\omega_3^2 = \cos\theta d\varphi$$

$\Omega_1^0 = d\omega_1^0 - (\ldots)$ ，( )中皆為 0

$$= d(\dot{a}(1-kr^2)^{-\frac{1}{2}}dr) = \ddot{a}(1-kr^2)^{-\frac{1}{2}}dt \wedge dr = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^1 = \Omega_0^1$$

$$\Omega_2^0 = \Omega_0^2 = d\omega_2^0 - (\omega_2^1 \wedge \omega_1^0 + \omega_2^3 \wedge \omega_3^0) = \ddot{a}rdt \wedge d\theta = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^2$$

Where $d\omega_2^0 = d(\dot{a}rd\theta) = \dot{a}\,dr \wedge d\theta + \ddot{a}\,rdt \wedge d\theta$

$\omega_2^1 \wedge \omega_1^0 = -(1-kr^2)^{\frac{1}{2}} d\theta \wedge \dot{a}(1-kr^2)^{-\frac{1}{2}} dr = \dot{a}\,dr \wedge d\theta$ ，

$\omega_2^3 \wedge \omega_3^0 = \cos d\varphi \wedge \dot{a}\,r\sin\theta d\varphi = 0$

$\Omega_3^0 = d\omega_3^0 - (\omega_3^1 \wedge \omega_1^0 + \omega_3^2 \wedge \omega_2^0)$

$= d(\dot{a}\,r\sin\theta d\varphi) - (-(1-kr^2)\sin\theta d\varphi \wedge \dot{a}(1-kr^2)^{-\frac{1}{2}} dr + (-\cos\theta d\varphi) \wedge \dot{a}\,rd\theta))$

$= \ddot{a}\,r\sin\theta dt \wedge d\varphi + \dot{a}\sin\theta dr \wedge d\varphi + \dot{a}\,r\cos\theta d\theta \wedge d\varphi + \dot{a}\sin\theta d\varphi \wedge dr - \dot{a}\,r\cos\theta d\theta \wedge d\varphi$

$= \frac{\ddot{a}}{a}\omega^0 \wedge \omega^3$

$\Omega_1^2 = d\omega_1^2 - (\omega_1^0 \wedge \omega_0^2 + \omega_1^3 \wedge \omega_3^2)$

$= d((1-kr^2)^{\frac{1}{2}} d\theta) - (\dot{a}(1-kr^2)^{-\frac{1}{2}} dr \wedge \dot{a}\,rd\theta + (1-kr^2)^{\frac{1}{2}} \sin\theta d\varphi \wedge \cos\theta d\varphi)$

$= \frac{1}{2}(1-kr^2)^{-\frac{1}{2}}(-2kr)dr \wedge d\theta - (\dot{a})^2 r(1-kr^2)^{-\frac{1}{2}} dr \wedge d\theta$

$= \frac{k}{a^2}\omega^2 \wedge \omega^1 + \frac{(\dot{a})^2}{a^2}\omega^2 \wedge \omega^1 = (\frac{k}{a^2} + \frac{(\dot{a})^2}{a^2})\omega^2 \wedge \omega^1$

同理

$\Omega_1^3 = -\Omega_3^1 = (\frac{k}{a^2} + \frac{(\dot{a})^2}{a^2})\omega^3 \wedge \omega^1$

$\Omega_2^3 = -\Omega_3^2 = (\frac{k}{a^2} + \frac{(\dot{a})^2}{a^2})\omega^3 \wedge \omega^2$

$R_{00} = R_{100}^1 + R_{200}^2 + R_{300}^3 = -(R_{010}^1 + R_{020}^2 + R_{030}^3) = -\frac{3\ddot{a}}{a}$

Where $R_{010}^1 = \Omega_0^1(E_0, E_1) = \frac{\ddot{a}}{a}$ ， $R_{020}^2 = \Omega_0^2(E_0, E_2) = \frac{\ddot{a}}{a}$ ， $R_{030}^3 = \Omega_0^3(E_0, E_3) = \frac{\ddot{a}}{a}$

$$R_{11}=R^{0}_{011}+R^{2}_{211}+R^{3}_{311}=-(R^{0}_{101}+R^{2}_{121}+R^{1}_{133})$$

$$=\frac{\ddot{a}}{a}+\frac{2(\dot{a})^2}{a^2}+\frac{2k}{a^2}$$

$R^{0}_{101}=-\frac{\ddot{a}}{a}$ ， $R^{2}_{121}=\Omega^{2}_{1}(E_1,E_2)=-(\frac{k}{a^2}+\frac{(\dot{a})^2}{a^2})$ ， $\mathrm{R}^{3}_{131}=\Omega^{3}_{1}(E_1,E_3)=-(\frac{k}{a^2}+\frac{(\dot{a})^2}{a^2})$

同理可求 $R_{22}=R_{33}=...=R_{11}$