  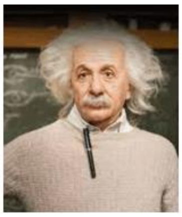 

1. Ricci flow 與 Poincare 猜想

   Hamilton's Ricci flow equation is $\frac{\partial}{\partial t} g_{ij} = -2R_{ij}$ ， $g(0) = g_0$

2. 愛因斯坦方程式 $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

3. 最優運算問題(里奇曲率的綜合運用)中都出現，

到底 Ricci 曲率的意義為何？

里奇曲率反應非歐幾何中體積的扭曲。

§ Ricci curvature and Scalar curvature

(1) Metric g

(2) $\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$

$$\nabla_X Y = \sum_i \left(XY^i + \sum_{j,k} \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial x^i}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

(3) $R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk} \Gamma^l_{im} - \sum_m \Gamma^m_{ik} \Gamma^l_{jm}$ ， $R_{ij} = R^k_{ikj}$

(4) $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ ， $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

(1) 3-sphere S^3 的 Riemannian tensor，Ricci tensor，Ricci scalar

(2) $I \times S^2$ 的 Ricci tensor (RG4102-2)

(3) RG4103RicciScalarforBH

(4) Exam2009

(a) Suppose that for some smooth function ρ ，we have $R_{ij} = \rho g_{ij}$ on the whole manifold M。Show that ρ is constant and $\rho = \frac{R}{n}$ ，n>2

(b) Let $(R^2, g(t))$ be a complete Riemannian surface with $g(t) = \frac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$

Show that

(1) In polar coordinates (r,θ) , we may rewrite

$$g(0) = ds^2 + \tanh^2 s d\theta^2, s = \log(r + \sqrt{1+r^2})$$

(2) The scalar curvature of $(R^2, g(0))$, $R_0 = \frac{4}{1+r^2}$

(3) Find 1-parameter group of conformal diffeomorphisms $\varphi_t : R^2 \to R^2$

Such that $g(t) = \varphi_t^* g(0)$

(5) Exam2015 $\mathrm{I} \times \mathrm{S}^2$

Consider the metric $g = A^2(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$ on $M = I \times S^2$
Where r is a local coordinates on $I \subset R$ and (θ, ϕ) are spherical local coordinates on S^2

(a) Compute the Ricci curvature and the scalar curvature of this metric

(b) What happens when $A(r) = \frac{1}{\sqrt{1-r^2}}$?

(c) What happens when $A(r) = \frac{1}{\sqrt{1+r^2}}$?

(d) For which fnction A® is the scalar curvature constant ?

(6) Exam2018

1.

(a) Suppose that (M,g) is a 3-dimensional Riemannian manifold which is Ricci flat 。
Does it have to be flat ?
If your answer is yes , give a proof 。 If not , give a counter-example 。
What if M is 4-dimensional Ricci flat manifold ?

(b) Suppose that (M,g) is a 3-dimensional Riemannian manifold which is Ricci flat 。
Does it have to be flat ?
If your answer is yes , give a proof 。 If not , give a counter-example 。
What if M is 4-dimentional Ricci flat manifold ? Explain the reason breifly 。

2. Let c be a non-negative conatant 。 Consider the following metric on R^n ,

$$g_c = \frac{\sum_{i=1}^{n}(dx^i)^2}{(1+\frac{c}{4}\sum_{i=1}^{n}(x^i)^2)^2}$$

(a) Calculate the sectional curvatures of (R^n, g_c)

For which values of c is (R^n, g_c) complete？Justify your answer。

(7) Exam2019 這裡有些參考資料

1. On R^3，consider the following metric

   $ds^2 = dx^2 + dy^2 + (dz + \sin zdx + \cos zdy)^2$

   (a) Calculate the Riemann curvatre tensor of ds^2

   (b) Denote the 1-form $dz + \sin zdx + \cos zdy$ by α。

   Can you find a regular surface Σ passing through the origin，and

   $T_p\Sigma \subset \ker(\alpha|_p)$ for every $p \in \Sigma$？Justify your answer。

   (c) Same question as (b) for the 1-form $\beta = dz + zdx$

   Can you find a regular surface Σ passing the origin，and $T_p\Sigma \subset \ker(\beta|_p)$

   for every $p \in \Sigma$？Justify your answer。

1. [Optimal Transport and curvature] by Cedric Villani https://cedricvillani.org/

2. https://profoundphysics.com/the-ricci-tensor/ 這裡有詳細的解說與例子：

The Ricci tensor represents how a volume in a curved space differs from a volume in Euclidean space。

In particular， the Ricci tensor measures how a volume between geodesics changes due to curvature。

In general relativity，the Ricci tensor represents volume changes due to gravitational tides。

Examples

1. 2-sphere S^2

   $ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$

   $$g_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix},\quad \Gamma^1_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{pmatrix},\quad \Gamma^2_{ij} = \begin{pmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{pmatrix}$$

the Ricci tensor $R_{ij} = \frac{g_{ij}}{r^2} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

2. Ricci tensor for the Schwarzschild metric

   The Schwarzschild metric is a solution of Einstein's field equations in a vacuum。

   In particular, it described the **spacetime around a spherically symmetric mass**。

   Now， since it is a vacuum solution， the energy-momentum tensor on the right-hand side of Einstein's equations is zero。

   $R_{\mu\nu} = 0$

3. Ricci tensor for the Kerr metric

4. Ricci tensor for the Roberson-Walker (FRW) metric

$$ds^2 = -dt^2 + \frac{a^2\,(t)}{1-kr^2}dr^2 + a^2\,(t)\,r^2 d\theta^2 + a^2\,(t)\,r^2 \sin^2\theta d\phi^2$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2\theta \end{pmatrix}$$

$$\Gamma^0_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{a\dot{a}}{c(1-kr^2)} & 0 & 0 \\ 0 & 0 & \frac{1}{c}a\dot{a}r^2 & 0 \\ 0 & 0 & 0 & \frac{1}{c}a\dot{a}r^2 \sin^2\theta \end{pmatrix}$$

$\dot{a} = \frac{da}{dt}$，and c is the speed of light

$$\Gamma^1_{\mu\nu} = \begin{pmatrix} 0 & \frac{\dot{a}}{ca} & 0 & 0 \\ \frac{\dot{a}}{ca} & \frac{kr}{1-kr^2} & 0 & 0 \\ 0 & 0 & -r\left(1-kr^2\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-kr^2\right) \end{pmatrix}$$

$$\Gamma^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{\dot{a}}{ca} & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ \frac{\dot{a}}{ca} & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix}$$

$$\Gamma^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \frac{\dot{a}}{ca} \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ \frac{\dot{a}}{ca} & \frac{1}{r} & \cot\theta & 0 \end{pmatrix}$$

Time component $R_{00} = -\frac{3}{c^2}\frac{\ddot{a}}{a}$, space component $R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2kc^2)\frac{g_{ij}}{a^2}$

$$R_{\mu\nu} = \begin{pmatrix} -\frac{3}{c^2}\frac{\ddot{a}}{a} & 0 & 0 & 0 \\ 0 & \frac{a\ddot{a}+2\dot{a}^2+kc^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & \left(a\ddot{a}+2\dot{a}^2+kc^2\right)r^2 & 0 \\ 0 & 0 & 0 & \left(a\ddot{a}+2\dot{a}^2+kc^2\right)r^2\sin^2\theta \end{pmatrix}$$

5. Ricci tensor for the Reissner-Nordstrom metric (a black hole)

$$g_{\mu\nu} = \begin{pmatrix} -\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right) & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$r_Q^2 = \frac{Q^2G}{4\pi\varepsilon_0 c^4} \quad \& \quad r_s = \frac{2GM}{c^2}$$

以下 略　　by Ville Hirvonen