§ 曲率

黎曼流形(M,g) 先給定度量，演算四個基本的式子：

(1) Christoffel symbol $\Gamma^i_{jk} = \frac{1}{2} g^{il} (\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})$

(2) 共變微分(covariant derivative of Y along X) $\nabla_X Y = \sum_i (XY^i + \sum_{jk} \Gamma^i_{jk} X^j Y^k)$

$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\sigma} V^\sigma$

(3) 測地線方程式 $\frac{d^2 x^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0$

(4) 黎曼張量 $R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk} \Gamma^l_{im} - \sum_m \Gamma^m_{ik} \Gamma^l_{jm}$

(對應 energy-momentum tensor)

(5) $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

(M,g)是一 Riemann 流形

$g: T_p M \times T_p M \to R$ 是 M 上的 metric

$ds^2 = g_{ij} dx^i dx^j$

在 R^n 上 $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \delta_{ij}$ 這時候是平直流形。

「黎曼空間是平直的$\Leftrightarrow R^j_{ijk} = 0$ 」

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \in \chi(M)$ is the torsion of connection ∇

$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]} Z$ is the curvature of the connection ∇，measure the deviation of the map $X \to \nabla_X$ from being a Lie algebra homomorphism。

(M, g) is said to be locally conformally flat if for $\forall p \in M$ ，there is a local coordinate system $\{x^i\}$ in a n. b. d. U of p such that $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = v\delta_{ij}$ for some function v

Schoen-Yau

If (M, g) is a simple、connected、locally conformally flat、complete Riemannian manifold，then exists a one to one conformal map of (M, g) into the standard sphere S^n

$\nabla_X e_j = \sum \omega^i_j(X)e_i$ ， ω^i_j are called connection forms， $\omega = [\omega^i_j]$ is called connection matrix。

$R(X,Y)e_j = \sum \Omega^i_j(X,Y)e_i$ ， Ω^i_j are called curvature forms， $\Omega = [\Omega^i_j]$ is called curvature matrix。 $\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^j_i$

If α, β are one forms，X,Y are vector fields，then

$(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$

$(d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])$

Prove $\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^j_i$

$\nabla_X \nabla_Y e_j = \nabla_X \sum_k (\omega^k_j(Y)e_k)$ definition of connection forms

$= \sum_k X\omega^k_j(Y)e_k + \sum_k \omega^k_j \nabla_X e_k$ Leibniz rule

$= \sum_i X\omega^i_j(Y)e_i + \sum_{j\ k} \omega^k_j(Y)\omega^i_k(X$

Interchanging X and Y gives $\nabla_Y \nabla_X e_j = \sum_i Y\omega^i_j(X)e_i + \sum_{i,k} \omega^k_j(X)\omega^i_k(Y)e_i$

Furthermore $\nabla_{[X,Y]} e_j = \sum_i \omega^i_j([X,Y]e_i$

Hence, in Einstein notation,

$$
\begin{aligned}
R(X,Y)e_j &= \nabla_X \nabla_Y e_j - \nabla_Y \nabla_X e_j - \nabla_{[X,Y]} e_j \\
&= (X\omega^i_j(Y) - Y\omega^i_j(X) - \omega^i_j([X,Y]))e_i \\
&\quad + (\omega^i_k(X)\omega^k_j(Y) - \omega^i_k(Y)\omega^k_j(X))e_i \\
&= d\omega^i_j(X,Y)e_i + \omega^i_k \wedge \omega^k_j(X,Y)e_i \qquad \text{(by (11.3) and (11.2))} \\
&= (d\omega^i_j + \omega^i_k \wedge \omega^k_j)(X,Y)e_i.
\end{aligned}
$$

Comparing this with the definition of the curvature form Ω^i_j gives

$$\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j.$$

黎曼曲率的脈絡

$$\Gamma_{ij}^{k}=\frac{1}{2}\sum_{l}g^{kl}(\frac{\partial g_{jl}}{\partial x^{i}}+\frac{\partial g_{il}}{\partial x^{j}}-\frac{\partial g_{ij}}{\partial x^{l}})$$

一. Riemann tensor curvature tensor Ricci curvature tensor

定義 $R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$

$R_{ijk}^{l}=\frac{\partial\Gamma_{jk}^{l}}{\partial x^{i}}-\frac{\partial\Gamma_{ik}^{l}}{\partial x^{j}}+\sum_{m}\Gamma_{jk}^{m}\Gamma_{im}^{l}-\sum_{m}\Gamma_{ik}^{m}\Gamma_{jm}^{l}$ Riemann tensor

$R_{ijkl}=\sum_{m}R_{ijk}^{m}g_{ml}$ curvature tensor

$R_{ij}=\sum_{k}R_{ikj}^{k}$ Ricci curvature tensor

$R=g^{\mu\nu}R_{\mu\nu}$ scalar curvature

$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ Eintein tensor

二. Cartan structure equations

$\{X_1,X_2,X_3,\ldots\}$ $\{\omega^1,\omega^2,\omega^3,.$ $\omega_j^k:=\sum_{i}\Gamma_{ij}^{k}\omega^{i}$ connection forms

(1) $d\omega^{i}=\sum_{j}\omega^{j}\wedge\omega_{j}^{i}$

(2) $dg_{ij}=\sum_{k}(g_{kj}\omega_{i}^{k}+g_{ki}\omega_{j}^{k})$

(3) $d\omega_{j}^{i}=\Omega_{j}^{i}+\sum_{k}\omega_{i}^{k}\wedge\omega_{k}^{j}$

The curvature R of a Riemannian manifold has the following properties：

1. $R(fX_1+gX_2,Y_1)=fR(X_1,Y_1)+gR(X_2,Y_1)$
   $R(X_1,fY_1+gY_2)=fR(X_1,Y_1)+gR(X_1,Y_2)$
2. $R(X,Y)(Z+W)=R(X,Y)Z+R(X,Y)W$
   $R(X,Y)fZ=fR(X,Y)Z$
3. R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0 Bianchi identity $R_{ijkl}+R_{iklj}+R_{iljk}=0$
4. $R_{ijkl}=-R_{ijlk}=-R_{jikl}=R_{klij}$

[DG06] p.157 有證明

1. Sectional curvature 截曲率

$K(\pi) := \langle R(e_1, e_2)e_2, e_1 \rangle$ where $\{e_1, e_2\}$ is an orthonormal basis of π

Prove $K(\pi) = \dfrac{\langle R(X,Y)Y, X \rangle}{|X|^2 |Y|^2 - \langle X, Y \rangle^2}$

2. Ricci tensor $Ric = \sum_{i,j} R_{ij} dx^i \otimes dx^j$ where $R_{ij} = \sum_k R_{kij}^k$

$$R_{ij} = \partial_k \Gamma_{ij}^k - \partial_j \Gamma_{ik}^k + \Gamma_{ij}^k \Gamma_{km}^m - \Gamma_{im}^k \Gamma_{jk}^m$$

3. Scalar curvature $S(p) = R := \sum_{i,j} g^{ij} R_{ij}$

4. Ricci flow $\dfrac{\partial g}{\partial t} = -Ric(g)$

5. Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \dfrac{1}{2} R g_{\mu\nu}$

習作

1. Let G be a Lie group with a bi-invariant metric <,>。Let $X, Y, Z \in \chi(M)$ be unit left invariant vector fields on G。
   (a) Show that $\nabla_X Y = \dfrac{1}{2}[X, Y]$
   (b) Conclude from (a) that $R(X,Y)Z = \dfrac{1}{4}[[X,Y],Z]$
   (c) Prove that if X and Y are orthonormal，the sectional curvature $K(\sigma)$ of G with respect to the plane σ generated by X and Y is given by $K(\sigma) = \dfrac{1}{4}\|[X,Y]\|^2$。Therefore the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is none-negative and is zero if and only if σ is generated by vectors X,Y which commute，that is [X,Y]=0

2. Let M be a Riemannian manifold with the following property：
   given any two points $p, q \in M$，the parallel transport from p to q does not depend on the curve that joints p to q。
   Prove that the curvature of M is identically zero，that is，for all $X, Y, Z \in \chi(M)$，R(X,Y)Z=0

3. Locally symmetric spaces

Let M be a Riemannian manifold。

M is locally symmetric space if $\nabla R = 0$，where R is the curvature tensor of M。

the curvature tensor of M. (The geometric significance of this condition will be given in Exercise 14 of Chap. 8).

a) Let M be a locally symmetric space and let $\gamma: [0, \ell) \to M$ be a geodesic of M. Let X, Y, Z be parallel vector fields along γ. Prove that $R(X, Y)Z$ is a parallel field along γ.

b) Prove that if M is locally symmetric, connected, and has dimension two, then M has constant sectional curvature.

c) Prove that if M has constant (sectional) curvature, then M is a locally symmetric space

4. Prove the second Bianchi Identity：

5. Schur's Theorem：
   Let M^n be a connected Riemannian manifold with $n \geq 3$。
   Suppose that M is isotropic，that is，for each $p \in M$，the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subset T_p M$。

   Prove that M has constant sectional curvature，that is，$K(p, \sigma)$ also does not depend on p

6. Prove that the scalar curvature K(p) at $p \in M$ is given by
   $K(p) = \dfrac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x) dS^{n-1}$，where ω_{n-1} is the area of the sphere S^{n-1} in $T_p M$ and dS^{n-1} is the area elements on S^{n-1}

7. A Riemannian manifold M^n is called an Einstein manifold if for all $X, Y \in \chi(M)$，$Ric(X, Y) = \lambda < X, Y >$，where $\lambda : M \to R$ is a real valued function。Prove that：

(a) If M^n is connected and Einstein with $n \geq 3$，then λ is constant on M

(b)If M^3 is a connected Einstein manifold then M^3 has constant sectional curvature

Hint 略