§ 曲率

四個基本的式子：

(1) Christoffel symbol

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right)$$

(2) 共變微分(covariant derivative of Y along X)

$$\nabla_X Y = \sum_i \left( X^j \nabla^j Y + \sum_k \Gamma^i_{jk} X^j Y^k \right)$$

(3) 测地线方程式

$$\frac{d^2 x^k}{dt^2} + \sum_{ij} \Gamma^k_{ij} \frac{dx^j}{dt} \frac{dx^i}{dt} = 0$$

(4) 黎曼张量$$R^l_{ijk} = \frac{\partial \Gamma^l_{ij}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk} \Gamma^l_{im} - \sum_m \Gamma^m_{ik} \Gamma^l_{jm}$$

黎曼 1854 年：「什麼時候空間平直？」

借助座標來描述 黎曼空間的座標什麼時候可以換成平直的新座標？

新座標該滿足的微分方程組，可否積分？Frobenius 定理回答這個問題。

根據 Frobenius 定理，引入黎曼(曲率)張量

(M,g)是→ Riemann 流形

g: T_p M \times T_p M \rightarrow R 是 M 上的 metric

$$ds^2 = g_{ij} dx^i dx^j$$

在 R^3 上，$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \delta_{ij}$$ 這時候是平直流形。

S^2 浸映(immersion)在 R^3 不會是一個平直流形？

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = [\nabla_X, \nabla_Y] Z$$

「 黎曼空間是平直的 ⇔ $$R^l_{ijk} = 0$$ 」

接著 利用測地線與 Jacobi 場去了解黎曼流形的整體樣貌，切入大域微分幾何的內核。

其間有許多花草花草 例如 共變微分(covariant derivative)、平行性(parallel)、
黎曼尺度(metric)、指數映射(exponential map)....。
(M, g) is said to be locally conformally flat if for \( \forall p \in M \), there is a local coordinate system \( \{x'\} \) in a n.b.d. U of p such that 
\[
g_{ij} = g\left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial x'}\right) = v \delta_{ij}
\]
for some function v.

Schoen-Yau
If (M, g) is a simple, connected, locally conformally flat, complete Riemannian manifold, then exists a one to one conformal map of (M, g) into the standard sphere \( S^n \).

黎曼曲率的誕生  1854
No open set of the 2-sphere \( S^2 \) with the standard metric is isometric to an open set of the Euclidean plane.
The geometric object that locally distinguishes these two Riemannian manifolds is the so-called curvature operator.

\[
\nabla_X e_j = \sum \omega_j^i(X)e_i \quad \omega_j^i \text{ are called connection forms}, \quad \omega = [\omega_j^i] \text{ is called connection matrix}.
\]

\[
R(X,Y)e_j = \sum \Omega_j^i(X,Y)e_i \quad \Omega_j^i \text{ are called curvature forms}, \quad \Omega = [\Omega_j^i] \text{ is called curvature matrix}.
\]

\[
\Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^i
\]

\[
R(X,Y) = \nabla_X Y - \nabla_Y X - \nabla_{[X,Y]}
\]

If \( \alpha, \beta \) are one forms, \( X,Y \) are vector fields, then
\[
(\alpha \wedge \beta)(X,Y) = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)
\]
\[
(d\alpha)(X,Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y])
\]

Prove \( \Omega_j^i = d\omega_j^i + \sum_k \omega_k^i \wedge \omega_j^i \)

\[
\nabla_X \nabla_Y e_j = \nabla_X \sum_k (\omega_j^k(Y)e_k) \quad \text{(definition of connection forms)}
\]
\[
= \sum_k X\omega_j^k(Y)e_k + \sum_k \omega_j^k(Y)\nabla_X e_k \quad \text{(Leibniz rule)}
\]
\[
= \sum_i X\omega_j^i(Y)e_i + \sum_{i,k} \omega_j^i(Y)\omega_k^j(X)e_i.
\]
Interchanging $X$ and $Y$ gives
\[
\nabla_Y \nabla_X e_j = \sum_i Y \omega^i_j(X) e_i + \sum_{i,k} \omega^i_j(X) \omega^k_j(Y) e_i.
\]

Furthermore,
\[
\nabla_{[X,Y]} e_j = \sum_i \omega^i_j([X,Y]) e_i.
\]

Hence, in Einstein notation,
\[
R(X,Y)^e_j = \nabla_X \nabla_Y^e_j - \nabla_Y \nabla_X^e_j - \nabla_{[X,Y]}^e_j
\]
\[
= \left( X \omega^i_j(Y) - Y \omega^i_j(X) - \omega^i_j([X,Y]) \right) e_i
\]
\[
+ (\omega^i_k(X) \omega^k_j(Y) - \omega^i_k(Y) \omega^k_j(X)) e_i
\]
\[
= d \omega^i_j(X,Y) e_i + \omega^i_k \wedge \omega^k_j(X,Y) e_i
\]
(by (11.3) and (11.2))
\[
= (d \omega^i_j + \omega^i_k \wedge \omega^k_j)(X,Y) e_i.
\]

Comparing this with the definition of the curvature form $\Omega^j$ gives
\[
\Omega^j = d \omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j.
\]

(4) Consider the usual local coordinates $(\theta, \varphi)$ in $S^2 \subset R^3$ defined the parameterization $\phi: (0, \pi) \times (0, 2\pi) \to R^3$ given by $X(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$

(a) Using these coordinates determine the expression of the Riemannian metric induced on $S^2$ by the Euclidean metric of $R^3$
\[
\frac{\partial}{\partial \theta} = \frac{\partial X}{\partial \theta} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)
\]
\[
\frac{\partial}{\partial \varphi} = \frac{\partial X}{\partial \varphi} = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0)
\]
\[
g_{\theta \theta} = \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \rangle = 1, \quad g_{\theta \varphi} = g_{\varphi \theta} = \langle \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi} \rangle = 0, \quad g_{\varphi \varphi} = \langle \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \rangle = \sin^2 \theta
\]
\[
\therefore g = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi
\]
(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates
\[
(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix}, \quad (g^{ij}) = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{\sin^2 \theta} \end{pmatrix}
\]
\[
\Gamma^\theta_{\varphi \varphi} = \frac{1}{2} \sum_{l=1}^2 g^{il} \left( \frac{\partial g_{el}}{\partial \varphi} + \frac{\partial g_{el}}{\partial \varphi} - \frac{\partial g_{el}}{\partial x_l} \right) = \frac{1}{2} \left( \frac{\partial \sin^2 \theta}{\partial \varphi} \right) = -\sin \theta \cos \theta
\]
\[
\text{同理} \Gamma^\varphi_{\theta \theta} = \Gamma^\varphi_{\varphi \theta} = \cot \theta
\]
(c) Show that the equator is the image of a geodesic.

Geodesic equation \( x^i + \sum_{j,k} \Gamma^i_{jk} x^j x^k = 0 \) for i=1, 2

得 \( \theta - \sin \theta \cos \theta \phi^2 = 0 \), \( \phi + 2 \cot \theta \theta \phi = 0 \)

赤道
\[
\begin{cases}
\theta(t) = \frac{\pi}{2} \\
\phi(t) = t
\end{cases}
\]

(d) Show that any rotation about an axis through the origin in \( R^3 \) induces an isometry of \( S^2 \).

(e) Show that the images of geodesics of \( S^2 \) are great circles.

(f) Find a geodesic triangle whose internal angles add up to \( \frac{3\pi}{2} \).

(g) Let \( c: R \rightarrow S^2 \) be given by \( c(t) = (\sin \theta_0 \cos t, \sin \theta_0 \sin t, \cos \theta_0) \) \( \theta_0 \in (0, \frac{\pi}{2}) \) (therefore c is not a geodesic). Let V be a vector field parallel along c such that \( V(0) = \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) \) is well defined at \( (\sin \theta_0, \cos \theta_0) \) by continuity.

Compute the angle by which V is rotated when it returns to the initial point.

(Remark: The angle you have computed is exactly the angle by which the oscillation plane of the Foucault pendulum rotates during a day in a place at latitude \( \frac{\pi}{2} - \theta_0 \), as it tries to remain fixed with respect to the stars on a rotating Earth)

(h) Use this result to prove that no open set \( U \subset S^2 \) is isometric to an open set \( W \subset R^2 \) with the Euclidean metric.

(i) 求 \( S^2 \) 的 Gauss curvature

\( g = A^2 (r) dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin \theta d\phi \otimes d\phi \) on \( M = I \times S^2 \) 計算 Ricci 張量
\((M, g)\)是一個黎曼流形

\[ X_1 = \frac{\partial}{\partial r}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \phi} \]

\(\langle X_1, X_1 \rangle = A(r)^2, \langle X_2, X_2 \rangle = r^2, \langle X_3, X_3 \rangle = r^2 \sin^2 \theta\) 所以取 orthonormal frames

為 \(E_1 = \frac{1}{A} \frac{\partial}{\partial r}, E_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, E_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}\)

An orthonormal coframes \(\omega^i = A(r)dr, \omega^2 = rd\theta, \omega^3 = r \sin \theta d\phi\)

由結構方程式 \(d\omega^i = \omega^j \wedge \omega_j^i\)

\(d\omega^1 = 0 = \omega^2 \wedge \omega^1_2 + \omega^3 \wedge \omega^1_3...\) (1)

\(d\omega^2 = dr \wedge d\theta = \omega^1 \wedge \omega^2_1 + \omega^3 \wedge \omega^2_3...\) (2)

\(d\omega^3 = \sin \theta dr \wedge d\theta + r \cos \theta d\theta \wedge d\phi = \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2...\) (3)

由(2) \(dr \wedge d\theta = A dr \wedge \omega^2_1 + r \sin \theta d\phi \wedge \omega^2_1\)，猜測 \(\omega^3_1 = \frac{1}{A} \, d\theta, r \sin \theta d\phi \wedge \omega^2_3 = 0\)

再由(1) 得 \(\omega^3 = \frac{\sin \theta}{A} \, d\phi\)，由(3) \(\omega^2_2 = \cos \theta d\phi\)

\[ \Omega^j_i = d\omega^i_j - \omega^i_j \wedge \omega^j_i \]

\[ \Omega^2_i = d\omega^2_i - \omega^2_i \wedge \omega^2_i = d\left(\frac{1}{A} \, d\theta\right) - \left(\frac{\sin \theta}{A} \, d\phi\right) \wedge \left(-\cos \theta d\phi\right) = -\frac{A'}{A^2} \, dr \wedge d\theta = \frac{A'}{A^3} \omega^1 \wedge \omega^2 \]

\[ \Omega^3_i = d\omega^3_i - \omega^3_i \wedge \omega^3_i = d\left(\frac{\sin \theta}{A} \, d\phi\right) - \left(\frac{1}{A} \, d\theta\right) \wedge \left(\cos \theta d\phi\right) \]

\[ = \left(-\frac{A'}{A^2} \sin \theta \, dr + \frac{\cos \theta}{A} \, d\theta \right) \wedge d\phi - \frac{\cos \theta}{A} \, d\theta \wedge d\phi \]

\[ = -\frac{A'}{A^2} \, d\theta \wedge d\phi = -\frac{A'}{A^3} \omega^1 \wedge \omega^3 \]

\[ \Omega^2_2 = d\omega^3_2 - \omega^3_2 \wedge \omega^3_2 = d(\cos \theta d\phi) - \left(-\frac{1}{A} \, d\theta\right) \wedge \left(\frac{\sin \theta}{A} \, d\phi\right) \]

\[ = (-\sin \theta \, d\theta \wedge d\phi) + \frac{\sin \theta}{A^2} \, d\theta \wedge d\phi \]

\[ = \frac{\sin \theta}{A^2} (\sin \theta) d\theta \wedge d\phi = \frac{1}{r^2} (\frac{1}{A^2} - 1) \omega^2 \wedge \omega^3 \]

\[ R_{ij}^k = \Omega^k_i (E_j, E_j) \quad R_{ij} = R_{kij} \]
\[ R^2_{i21} = \Omega^2_1(E_1, E_2) = -\frac{A'}{r^3}, \quad R^3_{i31} = \Omega^3_1(E_1, E_3) = -\frac{A'}{r^3}, \quad R^3_{232} = \Omega^3_2(E_2, E_3) = \frac{1}{r^2} \left(\frac{1}{A} - 1\right) \]

\[ R^i_{111} + R^i_{211} + R^i_{311} = -R^i_{121} - R^i_{131} = -\frac{A'}{r^3} \]

\[ R^i_{22} = \frac{A'}{r^3} - \frac{1}{r^2} \left(\frac{1}{A^2} - 1\right) \]

\[ R^i_{33} = \frac{A'}{r^3} - \frac{1}{r^2} \left(\frac{1}{A^2} - 1\right) \]

**Definition 3.1** A connection \( \nabla \) in a Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\) is said to be compatible with the metric if

\[ X \cdot \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \]

for all \( X, Y, Z \in \mathfrak{X}(M) \).

**Theorem 3.2** (Levi-Civita) If \((M, \langle \cdot, \cdot \rangle)\) is a Riemannian manifold then there exists a unique connection \( \nabla \) on \( M \) which is symmetric and compatible with \( \langle \cdot, \cdot \rangle \). In local coordinates \((x^1, \ldots, x^n)\), the Christoffel symbols for this connection are

\[ \Gamma^i_{jk} = \frac{1}{2} \sum_{l=1}^{n} g^{il} \left( \frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right) \quad (3.5) \]

where \((g^{ij}) = (g_{ij})^{-1}\).

黎曼曲率的脈絡

一. Affine connections \( \nabla \)

\[ X = \sum_i X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_i Y^i \frac{\partial}{\partial x^i} \]

then  \( \nabla_X Y = \sum_i (XY^i + \sum_{j,k} \Gamma^i_{jk} X^j Y^k) \frac{\partial}{\partial x^i} \)

\( \nabla_X Y \) : \( Y \) 沿 \( X \) 的方向導數 (covariant derivative)

If \( \nabla \) is symmetric then \( \nabla_X Y - \nabla_Y X = [X, Y] = \sum_i (XY^i - YX^i) \frac{\partial}{\partial x^i} \)

二. Parelle transport
三. Levi-Civita- connections

Riemannian manifold \((M, g)\) with metric \(g\) 相容

\[ X \cdot <Y, Z> = <\nabla_X Y, Z> \]

\[ X \cdot [Y, Z] = [\nabla_X Y, Z] + [Y, \nabla_X Z] \]

Then \( \Gamma^i_j = \frac{1}{2} \sum_k g^{ik} \left( \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} \right) \)

四. Riemann tensor curvature tensor Ricci curvature tensor

\[ R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^l_{jm} \Gamma^m_{ik} - \sum_m \Gamma^m_{ik} \Gamma^l_{jm} \] Riemann tensor

\[ R_{ijkl} = \sum_m R^m_{ijkl} g_{mi} \] curvature tensor

\[ R_y = \sum_k R^k_{ly} \] Ricci curvature tensor

五. Cartan structure equations

\{X_1, X_2, X_3, \ldots \} \{\omega^1, \omega^2, \omega^3, \ldots \} \omega^i = \sum_i \Gamma^i_j \omega^j \] connection forms

(1) \( d\omega^i = \sum_j \omega^i \wedge \omega^j \)

(2) \( dg_{ij} = \sum_k (g_{ik} \omega^k + g_{jk} \omega^i) \)

(3) \( d\omega^i = \Omega^i_j + \sum_k \omega^k \wedge \omega^i \)

\[ \nabla_X Y = \sum_j (Xb^i + \sum_i b^i \omega^j(X)) \frac{\partial}{\partial x^j} \]

定義

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \]

\[ R(X, Y, Z, W) = g(R(X, Y)Z, W) \]

\[ R_y = \sum_{i,j} R^i_{yj} dx^i \otimes dx^j \]

註：do Carmo 的定義差一個負號

2.1 DEFINITION. The curvature \( R \) of a Riemannian manifold \( M \) is a correspondence that associates to every pair \( X, Y \in \mathcal{X}(M) \) a mapping \( R(X, Y) : \mathcal{X}(M) \rightarrow \mathcal{X}(M) \) given by

\[ R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \quad Z \in \mathcal{X}(M), \]

where \( \nabla \) is the Riemannian connection of \( M \).
The curvature $R$ of a Riemannian manifold has the following properties:

1. $R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$
   \[ R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2) \]

   \[ R(X, Y)fZ = fR(X, Y)Z \]

3. $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ \text{ Bianchi identity}

[DG06] p.157 有證明

Proposition

Let $\sigma \in T_p M$ be a two-dimensional subspace of the tangent space $T_p M$ and let

$x, y \in \sigma$ be two linearly independent vectors. Then $K(x, y) = \frac{(x, y, x, y)}{|x \wedge y|^2}$ does not depend on the choice of the vectors $x, y \in \sigma$.

Do Carmo 的書 $(X,Y,Z,T) := \langle R(X,Y)Z,T \rangle$

3.2 Definition. Given a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_p M$, the real number $K(x, y) = K(\sigma)$, where $\{x, y\}$ is any basis of $\sigma$, is called the sectional curvature of $\sigma$ at $p$.

1. Sectional curvature 截曲率
   \[ K(\pi) := \langle R(e_1, e_2)e_1, e_1 \rangle \] where $\{e_1, e_2\}$ is an orthonormal basis of $\pi$

   Prove $K(\pi) = \frac{\langle R(X, Y)Y, X \rangle}{|X|^2|Y|^2 - \langle X, Y \rangle^2}$

2. Ricci tensor $\text{Ric} = \sum_{i,j} R_{ij} dx^i \otimes dx^j$ where $R_{ij} = \sum_k R_{ij}^k$

3. Scalar curvature $S(p) = R := \sum_{i,j} g^{ij} R_{ij}$

4. Ricci flow $\frac{\partial g}{\partial t} = -\text{Ric}(g)$

5. Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$
習作

1. Let $G$ be a Lie group with a bi-invariant metric $<,>$. Let $X, Y, Z \in \mathcal{X}(M)$ be unit left invariant vector fields on $G$.

   (a) Show that $\nabla_X Y = \frac{1}{2} [X, Y]

   (b) Conclude from (a) that $R(X, Y)Z = \frac{1}{4} [[X, Y], Z]

   (c) Prove that if $X$ and $Y$ are orthonormal, the sectional curvature $K(\sigma)$ of $G$ with respect to the plane $\sigma$ generated by $X$ and $Y$ is given by

   

   $K(\sigma) = \frac{1}{4} \left\| [X, Y] \right\|^2$

   Therefore the sectional curvature $K(\sigma)$ of a Lie group with bi-invariant metric is non-negative and is zero if and only if $\sigma$ is generated by vectors $X, Y$ which commute, that is $[X, Y] = 0

2. Let $X$ be a Killing field on a Riemannian manifold $M$.

   2. Let $X$ be a Killing field (See Exercise 5 of Chap. 3) on a Riemannian manifold $M$. Define a mapping $A_X : \mathcal{X}(M) \to \mathcal{X}(M)$ by $A_X(Z) = \nabla_Z X$, $Z \in \mathcal{X}(M)$. Consider the function $f : M \to \mathbb{R}$ given by $f(q) = \langle X, X \rangle_q$, $q \in M$. Let $p \in M$ be a critical point of $f$ (that is, $df_p = 0$). Prove that for any $Z \in \mathcal{X}(M)$ at $p$,

   a) $\langle A_X(Z), X \rangle(p) = 0.

   b) $\langle A_X(Z), A_X(Z) \rangle(p) = \frac{1}{2} Z_p(Z \langle X, X \rangle) + \langle R(X, Z)X, Z \rangle.

   Hint for (b): Put $S = \frac{1}{2} ZZ \langle X, X \rangle - \langle R(X, Z)X, Z \rangle$.

   Using the Killing equation $\langle \nabla_Z X, X \rangle + \langle \nabla_X Z, X \rangle = 0$ (cf. Exercise 5 of Chap. 3), we obtain

   $S = \langle \nabla_{[X,Z]} X, Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle - \langle \nabla_X \nabla_Z, Z \rangle$.

   Using the Killing equation again, we obtain

   $S = -\langle \nabla_Z X, \nabla_X Z \rangle + \langle \nabla_Z X, \nabla_Z X \rangle$

   $+ \langle \nabla_Z X, \nabla_X Z \rangle - \langle \nabla_X X, \nabla_Z Z \rangle$

   $= \langle \nabla_Z X, \nabla_Z X \rangle - \langle \nabla_X X, \nabla_Z Z \rangle$.

   Because of the Killing equation at $p$, $\nabla_X X(p) = 0$, and we conclude the assertion.
3. Let $M$ be a compact Riemannian manifold of even dimension whose sectional curvature is positive. Prove that every Killing field $X$ on $M$ has a singularity (there exists a $p \in M$ such that $X(p)=0$)

**Hint:** Let $f: M \to R$ be the function $f(q) = \langle X, X \rangle(q), q \in M$, and let $p \in M$ be a minimum point of $f$ (Cf. the previous Exercise). Suppose that $X(p) \neq 0$. Define a linear mapping $A: T_p M \to T_p M$ by $A(y) = A_X Y = \nabla_Y X$, where $Y$ is an extension of $y \in T_p M$. Let $E \subset T_p M$ be orthogonal to $X(p)$. Use the previous exercise to show that $A: E \to E$ is an anti-symmetric isomorphism. This implies that $\dim E = \dim M - 1$ is even, which is a contradiction; thus $X(p) = 0$.

4. Let $M$ be a Riemannian manifold with the following property: given any two points $p, q \in M$, the parallel transport from $p$ to $q$ does not depend on the curve that joins $p$ to $q$. Prove that the curvature of $M$ is identically zero, that is, for all $X, Y, Z \in \mathfrak{X}(M)$, $R(X, Y)Z = 0$

**Hint:** Consider a parametrized surface $f: U \subset \mathbb{R}^2 \to M$, where

$$U = \{(s, t) \in \mathbb{R}^2; -\varepsilon < t < 1 + \varepsilon, -\varepsilon < s < 1 + \varepsilon, \varepsilon > 0\}$$

and $f(s, 0) = f(0, 0)$, for all $s$. Let $V_0 \in T_{f(0,0)}(M)$ and define a field $V$ along $f$ by: $V(s, 0) = V_0$ and, if $t \neq 0$, $V(s, t)$ is the parallel transport of $V_0$ along the curve $t \to f(s, t)$. Then, from Lemma 4.1,

$$\frac{D}{\partial s} \frac{D}{\partial t} V = 0 = \frac{D}{\partial t} \frac{D}{\partial s} V + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V.$$

Since parallel transport does not depend on the curve chosen, $V(s, 1)$ is the parallel transport of $V(0, 1)$ along the curve $s \to f(s, 1)$, hence $\frac{D}{\partial s} V(s, 1) = 0$. Thus,

$$R_{f(0,1)}\left(\frac{\partial f}{\partial t}(0, 1), \frac{\partial f}{\partial s}(0, 1)\right)V(0, 1) = 0.$$

Use the arbitrariness of $f$ and $V_0$ to conclude what is required.
5. Let $\gamma: [0, l] \to M$ be a geodesic and $X \in \mathfrak{X}(M)$ be such that $X(\gamma(0)) = 0$. Show that $\nabla_{\gamma'} (R(\gamma', X)\gamma')(0) = (R(\gamma', X')\gamma')(0)$, where $X' = \frac{DX}{dt}$.

Hint Let $R$ be the curvature tensor of Example 5.2. Observe that, for all $Z \in \mathfrak{X}(M)$, and $t = 0$,

$$0 = (\nabla_{\gamma'} R)(\gamma', X, \gamma', Z)$$

$$= \frac{d}{dt} \langle R(\gamma', X)\gamma', Z \rangle - \langle R(\gamma', X')\gamma', Z \rangle - \langle R(\gamma', X)\gamma', Z' \rangle$$

$$= (\nabla_{\gamma'} (R(\gamma', X)\gamma'), Z) - \langle R(\gamma', X')\gamma', Z \rangle.$$

6. Locally symmetric spaces

Let $M$ be a Riemannian manifold. $M$ is locally symmetric if $\nabla R = 0$, where $R$ is the curvature tensor of $M$. (The geometric significance of this condition will be given in Exercise 14 of Chap. 8).

a) Let $M$ be a locally symmetric space and let $\gamma: [0, \ell] \to M$ be a geodesic of $M$. Let $X, Y, Z$ be parallel vector fields along $\gamma$. Prove that $R(X, Y)Z$ is a parallel field along $\gamma$.

b) Prove that if $M$ is locally symmetric, connected, and has dimension two, then $M$ has constant sectional curvature.

c) Prove that if $M$ has constant (sectional) curvature, then $M$ is a locally symmetric space.

7. Prove the second Bianchi Identity.

8. Schur’s Theorem:

Let $M^n$ be a connected Riemannian manifold with $n \geq 3$. Suppose that $M$ is isotropic, that is, for each $p \in M$, the sectional curvature $K(p, \sigma)$ does not depend on $\sigma \subset T_p M$. Prove that $M$ has constant sectional curvature, that is, $K(p, \sigma)$ also does not depend on $p$. 
Hint: Define a tensor \( R' \) of order 4 by
\[
R'(W, Z, X, Y) = \langle W, X \rangle \langle Z, Y \rangle - \langle Z, X \rangle \langle W, Y \rangle.
\]
If \( K(p, \sigma) = K \) does not depend on \( \sigma \), by Lemma 3.4, \( R = KR' \). Therefore, for all \( U \in \mathcal{X}(M) \), \( \nabla_u R = (UK)R' \). Using the 2nd Bianchi identity (see Exercise 7):
\[
\nabla R(W, Z, X, Y, U) + \nabla R(W, Z, Y, U, X) + \nabla R(W, Z, U, X, Y) = 0,
\]
we obtain, for all \( X, Y, W, Z, U \in \mathcal{X}(M) \),
\[
0 = (UK)\left(\langle W, X \rangle \langle Z, Y \rangle - \langle Z, X \rangle \langle W, Y \rangle\right) + (XK)\left(\langle W, Y \rangle \langle Z, U \rangle - \langle Z, Y \rangle \langle W, U \rangle\right) + (YK)\left(\langle W, U \rangle \langle Z, X \rangle - \langle Z, U \rangle \langle W, X \rangle\right).
\]
Fix \( p \in M \). Because \( n \geq 3 \), it is possible, fixing \( X \) at \( p \), to choose \( Y \) and \( Z \) at \( p \) such that \( \langle X, Y \rangle = \langle Y, Z \rangle = \langle Z, X \rangle = 0 \), \( \langle Z, Z \rangle = 1 \). Put \( U = Z \) at \( p \). The relation above yields, for all \( W \),
\[
((XK)Y - (YK)X, W) = 0.
\]
Since \( X \) and \( Y \) are linearly independent at \( p \), we conclude that \( XK = 0 \) for all \( X \in T_p M \). Thus \( K = \text{const.} \)

9. Prove that the scalar curvature \( K(p) \) at \( p \in M \) is given by
\[
K(p) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x) dS^{n-1}, \quad \text{where} \quad \omega_{n-1} \quad \text{is the area of the sphere} \quad S^{n-1} \quad \text{in} \quad T_p M \quad \text{and} \quad dS^{n-1} \quad \text{is the area elements on} \quad S^{n-1} \]
**Hint:** Use the following general argument on quadratic forms. Consider an orthonormal basis $e_1, \ldots, e_n$ in $T_pM$ such that if $x = \sum_{i=1}^{n} x_i e_i$,

$$Ric_p(x) = \sum \lambda_i x_i^2, \quad \lambda_i \text{ real.}$$

Because $|x| = 1$, the vector $(x_1, \ldots, x_n) = \nu$ is a unit normal vector on $S^{n-1}$. Denoting $V = (\lambda_1 x_1, \ldots, \lambda_n x_n)$, and using Stokes Theorem, we obtain

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} (\sum \lambda_i x_i^2) dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} (V, \nu) dS^{n-1}$$

$$= \frac{1}{\omega_{n-1}} \int_{B^n} \text{div} V dB^n,$$

where $B^n$ is the unit ball whose boundary is $S^{n-1} = \partial B^n$.

Noting that $\text{vol } B^n/\omega_n = 1/n$, we conclude that

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x) dS^{n-1} = \frac{1}{n} \text{div} V = \frac{\sum \lambda_i}{n}$$

$$= \frac{\sum Ric_p(e_i)}{n} = K(p).$$

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} (\sum \lambda_i x_i^2) dS^{n-1} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} (V, \nu) dS^{n-1}$$

$$= \frac{1}{\omega_{n-1}} \int_{B^n} \text{div} V dB^n,$$

where $B^n$ is the unit ball whose boundary is $S^{n-1} = \partial B^n$.

Noting that $\text{vol } B^n/\omega_n = 1/n$, we conclude that

$$\frac{1}{\omega_{n-1}} \int_{S^{n-1}} Ric_p(x) dS^{n-1} = \frac{1}{n} \text{div} V = \frac{\sum \lambda_i}{n}$$

$$= \frac{\sum Ric_p(e_i)}{n} = K(p).$$
10. A Riemannian manifold $M^n$ is called an Einstein manifold if for all $X,Y \in \mathcal{X}(M)$, $\text{Ric}(X,Y) = \lambda \left< X,Y \right>$, where $\lambda : M \rightarrow \mathbb{R}$ is a real valued function. Prove that:

(a) If $M^n$ is connected and Einstein with $n \geq 3$, then $\lambda$ is constant on $M$.
(b) If $M^3$ is a connected Einstein manifold then $M^3$ has constant sectional curvature.

Hint 略

[DG07] by Sean Carroll  第三章  Curvature

Q:
Given a restriction on the curvature of a Riemannian manifold, what topological conditions follow?
1. Myers theorem:
2. Cartan-Hadamard theorem:

§ 3-1 Overview 有四個式子:
1. Christoffel Symbol
2. 聯絡 $\nabla \lambda Y$
3. geodesic equation
4. Riemann tensor $R^{i}_{\mu \nu \lambda}$

§ 3-2 Covariant Derivatives $\nabla$
流形的 metric 唯一決定一個聯絡 共變微分讓流形上的變換與座標無關(張量)

對於 one-form 的 covariant derivative 為 $\nabla_{\mu} \omega_{\nu} = \partial_{\mu} \omega_{\nu} - \Gamma_{\mu \nu}^{\lambda} \omega_{\lambda}$

There is exactly one torsion-free connection on a given manifold that is compatible with some given metric on that manifold.
(M, $g$) with connections is called Riemannian geometry. When the metric has a Lorentzian signature it called pseudo-Riemannian.

Metric compatibility: $\nabla_{\mu} g_{\mu \nu} = 0$
§ 3-3 Parallel Transport and Geodesics

The concept of moving a vector along a path, keeping constant all the while, is known as parallel transport.

We then define parallel transport of the tensor \( T \) along the path \( x^\mu(\lambda) \) to be the requirement that the covariant derivative of \( T \) along the path vanishes:

\[
\left( \frac{D}{d\lambda} T \right)_{\mu_1\mu_2\ldots\mu_k}^{\nu_1\nu_2\ldots\nu_l} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma T_{\mu_1\mu_2\ldots\mu_k}^{\nu_1\nu_2\ldots\nu_l} = 0. \tag{3.39}
\]

This is a well-defined tensor equation (since both the tangent vector \( dx^\mu/d\lambda \) and the covariant derivative \( \nabla T \) are tensors), known as the equation of parallel transport. For a vector it takes the form

\[
\frac{d}{d\lambda} V^\mu + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} V^\rho = 0. \tag{3.40}
\]

The notion of parallel transport is obviously dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, the metric is always parallel transported with respect to it:

\[
\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0. \tag{3.41}
\]

It follows that the inner product of two parallel-transported vectors is preserved. That is, if \( V^\mu \) and \( W^\nu \) are parallel-transported along a curve \( x^\sigma(\lambda) \), we have

\[
\frac{D}{d\lambda} (g_{\mu\nu} V^\mu W^\nu) = \left( \frac{D}{d\lambda} g_{\mu\nu} \right) V^\mu W^\nu + g_{\mu\nu} \left( \frac{D}{d\lambda} V^\mu \right) W^\nu + g_{\mu\nu} V^\mu \left( \frac{D}{d\lambda} W^\nu \right) = 0. \tag{3.42}
\]

This means that parallel transport with respect to a metric-compatible connection preserves the norm of vectors, the sense of orthogonality, and so on.

A geodesic is a curve along which the tangent vector is parallel transported.

\[
\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = 0 \quad \text{or alternatively} \quad \frac{d^2 x^k}{dt^2} + \sum_j \Gamma^k_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} = 0
\]

Geodesic 的另一個定義(最短距離)是 \( \tau = \int \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda \) 用到變分法

§ 3-4 Properties of Geodesics
§ 3-5 The Expanding Universe Revisited
§ 3-6 The Riemann Curvature Tensor
§ 3-7 Properties of Riemann Tensor
§ 3-8 Symmetries and Killing Vectors
§ 3-9 Maximally Symmetric Spaces
§ 3-10 Geodesic Deviation

後面有習作