§ S^2 & $I\times S^2$

[An Introduction to Riemannian Geometry] Ex 3.3 p.106 (4)

(4) Consider the usual local coordinates (θ,φ) in $S^2\subset\mathbb{R}^3$ defined by the parameterization $\phi:(0,\pi)\times(0,2\pi)\to\mathbb{R}^3$ given by

$$\phi(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta).$$

(a) Using these coordinates, determine the expression of the Riemannian metric induced on S^2 by the Euclidean metric of $\mathbb{R}^3$.
(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.
(c) Show that the equator is the image of a geodesic.
(d) Show that any rotation about an axis through the origin in $\mathbb{R}^3$ induces an isometry of S^2.
(e) Show that the images of geodesics of S^2 are great circles.
(f) Find a **geodesic triangle** (i.e. a triangle whose sides are images of geodesics) whose internal angles add up to $\frac{3\pi}{2}$.
(g) Let $c:\mathbb{R}\to S^2$ be given by $c(t)=(\sin\theta_0\cos t,\sin\theta_0\sin t,\cos\theta_0)$, where $\theta_0\in\left(0,\frac{\pi}{2}\right)$ (therefore c is not a geodesic). Let V be a vector field parallel along c such that $V(0)=\frac{\partial}{\partial\theta}$ ($\frac{\partial}{\partial\theta}$ is well defined at $(\sin\theta_0,0,\cos\theta_0)$ by continuity). Compute the angle by which V is rotated when it returns to the initial point. (**Remark:** The angle you have computed is exactly the angle by which the oscillation plane of the **Foucault pendulum** rotates during a day in a place at latitude $\frac{\pi}{2}-\theta_0$, as it tries to remain fixed with respect to the stars on a rotating Earth).
(h) Use this result to prove that no open set $U\subset S^2$ is isometric to an open set $W\subset\mathbb{R}^2$ with the Euclidean metric.
(i) Given a geodesic $c:\mathbb{R}\to\mathbb{R}^2$ of $\mathbb{R}^2$ with the Euclidean metric and a point $p\notin c(\mathbb{R})$, there exists a unique geodesic $\tilde{c}:\mathbb{R}\to\mathbb{R}^2$ (up to reparameterization) such that $p\in\tilde{c}(\mathbb{R})$ and $c(\mathbb{R})\cap\tilde{c}(\mathbb{R})=\varnothing$ (**parallel postulate**). Is this true in S^2?

$\varphi:M\to N$ is an immersion。(N, g) is a Riemannian manifold，then φ^*g is a Riemannian metric in M induced by φ

(a) $\phi:S^2\to R^3$, $\phi:(0,\pi)\times(0,2\pi)\to R^3$

Then $\phi^*g=d\theta^2+\sin^2\theta d\varphi^2$ is the induced metric on S^2

(R^3,g) $g=(dx)^2+(dy)^2+(dz)^2$

$(S^2,\tilde{g})$，$\tilde{g}=\varphi^*g$

$$\frac{\partial}{\partial\theta}=\frac{\partial\phi}{\partial\theta}=(\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta)$$

$$\frac{\partial}{\partial \varphi} = \frac{\partial \phi}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$$

$$\tilde{g}_{11} = g_{\theta\theta} = <\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}> = 1$$ ，

$$\tilde{g}_{12} = \tilde{g}_{21} = g_{\theta\varphi} = g_{\varphi\theta} = <\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi}> = 0$$ ， $$\tilde{g}_{22} = g_{\varphi\varphi} = <\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi}> = \sin^2\theta$$

$$\therefore \varphi^* g = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi$$

Or ，on S^2 $x = \sin\theta\cos\varphi, y = \sin\theta\sin\varphi, z = \cos\theta$

$$dx = \cos\theta\cos\varphi dx - \sin\theta\sin\varphi d\varphi$$
$$dy = \cos\theta\sin\varphi d\theta + \sin\theta\cos\varphi d\varphi$$
$$dz = -\sin\theta d\theta$$

Then $dx^2 + dy^2 + dz^2 = \ldots = d\theta^2 + \sin^2\theta d\varphi^2$

$(S^2, h) \xrightarrow{\phi} (R^3, g)$ ， $g = dx^2 + dy^2 + dz^2, h = d\theta^2 + \sin^2\theta d\varphi^2$

$$\phi^* g = d\theta^2 + \sin^2\theta d\varphi^2 = h$$

[RG3103InducedMetric]

(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates

$$\left(g_{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$ ， $$\left(g^{ij}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{pmatrix}$$

$$\Gamma^{\theta}_{\varphi\varphi} = \frac{1}{2}\sum_{l=1}^{2} g^{\theta l}\left(\frac{\partial g_{\varphi l}}{\partial\varphi} + \frac{\partial g_{\varphi l}}{\partial\varphi} - \frac{\partial g_{\varphi\varphi}}{\partial x^l}\right) = \ldots = \frac{1}{2}\left(-\frac{\partial \sin^2\theta}{\partial\theta}\right) = -\sin\theta\cos\theta$$

同理 $\Gamma^{\varphi}_{\theta\varphi} = \Gamma^{\varphi}_{\varphi\theta} = \cot\theta$

Or let $L = (\dot{\theta})^2 + \sin^2\theta(\dot{\phi})^2$ ，consider the Euler equation

$$\frac{\partial L}{\partial\theta} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \Rightarrow 2\sin\theta\cos\theta(\dot{\phi})^2 - \frac{d}{d\tau}(2\dot{\theta}) = 0$$

$\ddot{\theta} - \sin\theta\cos\theta(\dot{\phi})^2 = 0$ imply $\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$

$$\frac{\partial L}{\partial\phi} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = 0 \Rightarrow \frac{d}{d\tau}(\sin^2\theta \times (2\dot{\phi})) = 0$$

$2\sin\theta\cos\theta\dot{\theta}\dot{\phi} + \sin^2\theta\ddot{\phi} = 0$ $\ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0$ imply $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$

(c) Show that the equator is the image of a geodesic

Geodesic equation $\ddot{x}^i + \sum_{j,k} \Gamma^i_{jk} \dot{x}^j \cdot \dot{x}^k = 0$ for i=1，2

得 $\ddot{\theta} - \sin\theta\cos\theta(\dot{\phi})^2 = 0$ ， $\ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0$

赤道 $\begin{cases} \theta(t) = \dfrac{\pi}{2} \\ \phi(t) = t \end{cases}$

(d) Show that any rotation about an axis through the origin in R^3 induces an isometry of S^2

Any rotation about an axis through the origin in R^3 is an isometry of R^3 which preserves S^2。Since we are considering the metric in S^2 induced by the Euclidean metric on R^3，it is clear that such a rotation will determine an isometry of S^2。

(e) Show that the images of geodesics of S^2 are great circles

Given a point $p \in S^2$ and a vector $v \in T_pS^2$, there exists a rotation $R : \mathbb{R}^3 \to \mathbb{R}^3$ such that $R(p) = (1,0,0)$ and $R(v) = (0,1,0)$. The geodesic with these initial conditions is clearly the curve c given in coordinates by $\hat{c}(t) = (\theta(t), \varphi(t)) = \left(\frac{\pi}{2}, t\right)$, whose image is the equator. By Exercise 3.3(3), the geodesic with initial condition $v \in T_pS^2$ must be $R^{-1} \circ c$. Since the image of c is the intersection of S^2 with the plane $z = 0$, the image of $R^{-1} \circ c$ is the intersection of S^2 with some plane through the origin, i.e. a great circle.

(f) Find a geodesic triangle whose internal angles add up to $\dfrac{3\pi}{2}$

For exa,ple the triangle with vertices (1,0,0)，(0,1,0)，(0,0,1)

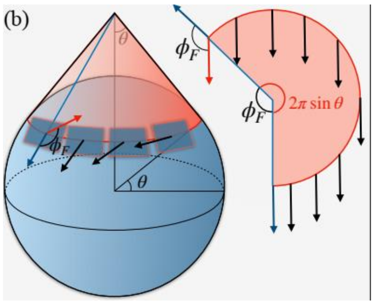
(g) Let $c : R \to S^2$ be given by $c(t) = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0)$，where

$\theta_0 \in (0, \frac{\pi}{2})$ (therefore c is not a geodesic)。Let V be a vector field parallel along c

such that $V(0) = \frac{\partial}{\partial\theta}(\frac{\partial}{\partial\theta}$ is well defined at $(\sin\theta_0, \cos\theta_0)$ by continuity)。

Compute the angle by which V is rotated when it returns to the initial point。

[RG3304Foucault]



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[From the geometry of Foucault pendulum to the topology of planetary waves]

The metric on S^2 is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ and

$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$，$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$

一個 vector field V^{μ} 沿曲線 $x^{\mu}(\lambda)$ 平行移動則 $\dfrac{d}{d\lambda}V^{\mu} + \Gamma^{\mu}_{ij}\dfrac{dx^i}{d\lambda}V^j = 0$

Along the curve c，$\theta = \theta_0$ is a constant，$\phi = t$ then $\dot{\theta} = 0, \dot{\phi} = \dfrac{d\phi}{dt} = 1$

The equations for paralleltransport are $\dot{V}^i + \sum_{j,k}\Gamma^i_{jk}\dot{x}^j V^k = 0$

$\dot{V}^{\theta} + \sum_{ij}\Gamma^{\theta}_{ij}\dot{x}^i V^j = 0$，$\dot{V}^{\theta} + \Gamma^{\theta}_{\phi\phi}\dot{\phi}V^{\phi} = 0$，$\dot{V}^{\theta} - \sin\theta_0\cos\theta_0 V^{\phi} = 0 \ldots (1)$

$\dot{V}^{\varphi} + \sum_{i,j}\Gamma^{\varphi}_{ij}\dot{x}^i V^j = 0$，$\dot{V}^{\phi} + \Gamma^{\phi}_{\theta\phi}\dot{\theta}V^{\phi} + \Gamma^{\phi}_{\phi\theta}\dot{\phi}V^{\theta} = 0$，$\dot{V}^{\phi} + \cot\theta_0 V^{\theta} = 0 \ldots (2)$

Since $\dot{\theta} = 0, \dot{\phi} = 1$

(1)式兩邊對 t 微分，$\ddot{V}^{\theta} = \sin\theta_0\cos\theta_0\dot{V}^{\phi} = -\sin\theta_0\cos\theta_0\cot\theta_0 V^{\theta} = -\cos^2\theta_0 V^{\theta}$

解此微分方程，$V^{\theta} = A\cos(t\cos\theta_0) + B\sin(t\cos\theta_0)$

$$V^{\phi} = \frac{1}{\sin\theta_0\cos\theta_0}\dot{V}^{\theta} = \frac{1}{\sin\theta_0\cos\theta_0}(-\cos\theta_0 A\sin(t\cos\theta_0) + \cos\theta_0 B\cos(t\cos\theta_0))$$

$$V(0) = \frac{\partial}{\partial\theta} = (V^{\theta}(0), V^{\phi}(0)) = (1,0) \Rightarrow V^{\theta}(0) = 1, V^{\phi}(0) = 0$$

Imply A=1、B=0，so $V^{\theta} = \cos(t\cos\theta_0), V^{\phi} = -\dfrac{1}{\sin\theta_0}\sin(t\cos\theta_0)$

Note that $|V| = g_{\mu\nu}V^{\mu}V^{\nu} = V^{\theta}V^{\theta} + \sin^2\theta_0 V^{\phi}V^{\phi} = 1$

假設 $V(0), V(2\pi)$ 的夾角為 α，

$$V(0) = \frac{\partial}{\partial\theta}, V(2\pi) = \cos(2\pi(\cos\theta_0))\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta_0}\sin(2\pi(\cos\theta_0))\frac{\partial}{\partial\phi}$$

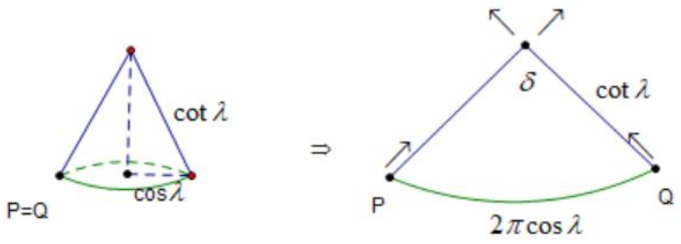
則$\cos\alpha = \langle V(0), V(2\pi)\rangle = \cos(2\pi\cos\theta_0)$

所以$\alpha = 2\pi\cos\theta_0$或$2\pi(1-\cos\theta_0)$

若傅科擺位於北緯λ度($\lambda = \frac{\pi}{2} - \theta_0$)， 則擺動方向每 24 小時轉動$2\pi\sin\lambda$

參考 [大域微分幾何]p.156~162

Do Carmo 第二章習作 p.58



$2\pi\cos\lambda = \cot\lambda \times \delta \Rightarrow \delta = 2\pi\sin\lambda$

(h) Use this result to prove that no open set $U \subset S^2$ is isometric to an open set $W \subset R^2$ with the Euclidean metric

Using the fact that any point on S^2 can be carried to (0, 0, 1) by an appropriate isometry, we just have to show that no open neighborhood $U \subset S^2$ of (1, 0, 0) is isometric to an open set $V \subset \mathbb{R}^2$ with the Euclidean metric. Now any such neighborhood contains the image of a curve $c(t)$ as given in (g) (for $\theta_0 > 0$ sufficiently small). If U were isometric to W, the Levi–Civita connection on U would be the trivial connection, and hence the parallel vector field $V(t)$ in (g) would satisfy $V(0) = V(2\pi)$. Since this is not true for any $\theta_0 \in (0, \frac{\pi}{2})$, U cannot be isometric to W.

(i) 求S^2的 Gauss curvature

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \text{，} \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

$$R^{\theta}_{\phi\theta\phi} = \partial_\theta\Gamma^{\theta}_{\phi\phi} - \partial_\phi\Gamma^{\theta}_{\theta\phi} + \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} - \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi}$$

$$= (\sin^2\theta - \cos^2\theta) - 0 + 0 - (-\sin\theta\cos\theta)(\cot\theta) = \sin^2\theta$$

$$R_{\theta\phi\theta\phi} = g_{\theta\lambda}R^{\lambda}_{\phi\theta\phi} = g_{\theta\theta}R^{\theta}_{\phi\theta\phi} = a^2\sin^2\theta$$

All the components of the Riemann tensor either vanish or are related to this one by symmetry。

The Ricci tensor $R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu}$

$$R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1$$

$$R_{\theta\phi} = R_{\phi\theta} = 0$$

$$R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2\theta$$

The Ricci scalar

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2}$$

p.139 Ex2.8 (6)

(6) Consider the metric

$$g = A^2(r) dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta \, d\varphi \otimes d\varphi$$

on $M = I \times S^2$, where r is a local coordinate on $I \subset \mathbb{R}$ and (θ, φ) are spherical local coordinates on S^2.

(a) Compute the Ricci tensor and the scalar curvature of this metric.

(b) What happens when $A(r) = (1 - r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to S^3)?

(c) And when $A(r) = (1 + r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to the **hyperbolic 3-space**)?

(d) For which functions $A(r)$ is the scalar curvature constant?

(M, g)是一個黎曼流形

$$X_1 = \frac{\partial}{\partial r}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \varphi}$$

$< X_1, X_1 >= A(r)^2, < X_2, X_2 >= r^2, < X_3, X_3 >= r^2 \sin^2\theta$，所以取 orthonormal frames

為 $E_1 = \frac{1}{A}\frac{\partial}{\partial r}, E_2 = \frac{1}{r}\frac{\partial}{\partial \theta}, E_3 = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$

An orthonormal coframes $\omega^1 = A(r)dr, \omega^2 = r d\theta, \omega^3 = r\sin\theta d\varphi$

由結構方程式 $d\omega^i = \omega^j \wedge \omega_j^i$

$$d\omega^1 = 0$$

$$d\omega^2 = dr \wedge d\theta = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 = Adr \wedge \omega_1^2$$

$$\therefore \omega_1^2 = \frac{1}{A} d\theta$$

$$d\omega^3 = \sin\theta dr \wedge d\theta + r\cos\theta d\theta \wedge d\varphi = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3$$

$$= Adr \wedge \omega_1^3 + rd\theta \wedge \omega_2^3 \text{ , } \therefore \omega_1^3 = \frac{\sin\theta}{A} d\varphi, \omega_2^3 = \cos\theta d\varphi$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = d(\frac{1}{A} d\theta) - (\frac{\sin\theta}{A} d\varphi) \wedge (-\cos\theta d\varphi) = \frac{-A'}{A^2} dr \wedge d\theta = \frac{-A'}{rA^3} \omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^2 - \omega_1^2 \wedge \omega_2^3 = d(\frac{\sin\theta}{A} d\varphi) - (\frac{1}{A} d\theta) \wedge (\cos\theta d\varphi)$$

$$= (\frac{-A'\sin\theta}{A^2} dr + \frac{\cos\theta}{A} d\theta) \wedge d\varphi - \frac{\cos\theta}{A} d\theta \wedge d\varphi$$

$$= \frac{-A'\sin\theta}{A^2} d\theta \wedge d\varphi = \frac{-A'}{rA^3} \omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = d(\cos\theta d\varphi) - (-\frac{1}{A} d\theta) \wedge (\frac{\sin\theta}{A} d\varphi)$$

$$= (-\sin\theta d\theta \wedge d\varphi) + \frac{\sin\theta}{A^2} d\theta \wedge d\varphi$$

$$= (\frac{\sin\theta}{A^2} - \sin\theta) d\theta \wedge d\varphi = \frac{1}{r^2}(\frac{1}{A^2} - 1)\omega^2 \wedge \omega^3$$

$$R_{iji}^j = \Omega_i^j(E_i, E_j) \text{ , } R_{ij} = R_{kij}^k$$

$$R_{121}^2 = \Omega_1^2(E_1, E_2) = \frac{-A'}{rA^3} \text{ , } R_{131}^3 = \Omega_1^3(E_1, E_3) = \frac{-A'}{rA^3} \text{ , } R_{232}^3 = \Omega_2^3(E_2, E_3) = \frac{1}{r^2}(\frac{1}{A} - 1)$$

$$R_{11} = R_{111}^1 + R_{211}^2 + R_{311}^3 = -R_{121}^2 - R_{131}^3 = \frac{2A'}{rA^3}$$

$$R_{22} = R_{122}^1 + R_{222}^2 + R_{322}^3 = -R_{121}^2 - R_{232}^3 = \frac{A'}{rA^3} - \frac{1}{r^2}(\frac{1}{A^2} - 1)$$

$$R_{33} = R_{133}^1 + R_{233}^2 + R_{333}^3 = -R_{131}^3 - R_{232}^3 = \frac{A'}{rA^3} - \frac{1}{r^2}(\frac{1}{A^2} - 1)$$

Scalar curvature $R = R_{11} + R_{22} + R_{33} = \frac{4A'}{rA^3} - \frac{2}{r}(\frac{1}{A^2} - 1)$