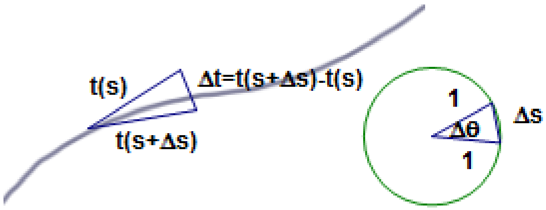
§ 古典微分幾何中的曲率



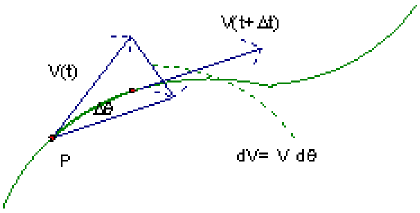
曲率向量

$\kappa=\lim_{\Delta s\to 0}\frac{t(s+\Delta s)-t(s)}{\Delta s}=\frac{dt}{ds}$ (單位切向量對弧長的變化率)

曲率= $\kappa=|\kappa|=\frac{d\theta}{ds}$

$|t(s+\Delta s)-t(s)|\approx\Delta s=\Delta\theta$

一質點沿一曲線 $\alpha(t)$ 作等速率運動，則加速度



$$|\vec{a}|=\lim_{\Delta t\to 0}\frac{V(t+\Delta t)-V(t)}{\Delta t}=\frac{dV}{dt}=\frac{DV}{ds}\cdot\frac{ds}{dt}=|V|\frac{d\theta}{ds}\cdot\frac{ds}{dt}=\kappa V^2$$

說明

(1) $s(t)=\int_{t_0}^{t}\left|\frac{d\alpha}{dt}\right|dt$，則 $\frac{ds}{dt}=\left|\frac{d\alpha}{dt}\right|=V(t)$

(2) 對等速圓周運動，因為 $\kappa=\frac{1}{R}$，所以 $|\vec{a}|=\frac{V^2}{R}$

$X'=\frac{dX}{ds}$，$\dot{X}=\frac{dX}{dt}$，則 $\kappa^2=X''\cdot X''$

因為 $X'\perp X''$ 且 $|X'|=1$，所以 $\kappa=|X'\wedge X''|=\frac{|\dot{X}\wedge\ddot{X}|}{|\dot{X}|^3}$

(證明在最後面)

◇習作◇

1. 求 y=f(x)的曲率函數
2. 求螺線(helix)X=(acost,asint,bt)的曲率
3. 橢圓 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 的曲率函數
4. $\alpha(t)=(r\cos t, r\sin t, 0)$，求 $\int_\alpha \kappa ds=$

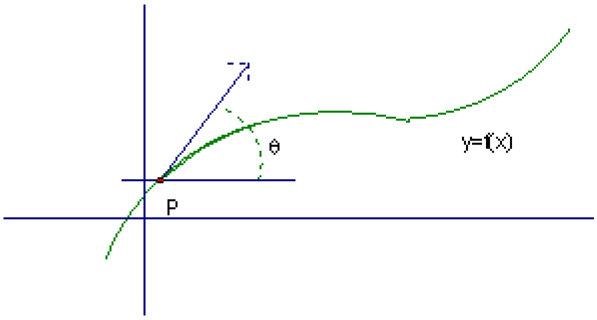
◇解答◇

1.求 y=f(x)的曲率函數

弧長 $ds^2=dx^2+dy^2$

$$\frac{ds}{dx}=\sqrt{1+(\frac{dy}{dx})^2}=\sqrt{1+(f'(x))^2}$$

所以 $\frac{dx}{ds}=\frac{1}{\sqrt{1+(f'(x))^2}}$ ......(1)



$f'(x)=\tan\theta$，所以 $f''(x)=\frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{dx}=\sec^2\theta\cdot\frac{d\theta}{dx}=(1+(f'(x))^2)\frac{d\theta}{dx}$

$\frac{d\theta}{dx}=\frac{f''(x)}{1+(f'(x))^2}$ ......(2)，由(1)(2)

$$\kappa=\frac{d\theta}{ds}=\frac{d\theta}{dx}\cdot\frac{dx}{ds}=\frac{f''(x)}{[1+(f'(x)^2]^{\frac{3}{2}}}$$

如果用弧長做參數，在反曲點 $f'(s)=0, f''(s)=0$，則 $\kappa=|f''(s)|=0$

2. $\kappa=\frac{a}{a^2+b^2}$

3. $X=(a\cos\theta, b\sin\theta)$，$\dot{X}=(-a\sin\theta, b\cos\theta)$，$\ddot{X}=(-a\cos\theta, -b\sin\theta)$

$\left|\dot{X}\wedge\ddot{X}\right|=ab$，所以$\kappa^2=\dfrac{\left|\dot{X}\times\ddot{X}\right|^2}{(\dot{X}\cdot\dot{X})^3}$ ，$\kappa=\dfrac{ab}{(a^2\sin^2\theta+b^2\cos^2\theta)^{\frac{3}{2}}}$

在短軸端點$\theta=\dfrac{\pi}{2}$，$\kappa=\dfrac{b}{a^2}$

$f(\theta)=a^2\sin^2\theta+b^2\cos^2\theta=b^2+(a^2-b^2)\sin^2\theta$在$\theta=0,\dfrac{\pi}{2}$產生極值

所以橢圓的″頂點″（$\dfrac{d\kappa}{ds}=0$的地方）在四個端點產生。

這是四頂點定理的一個例子。

4．$\alpha(t)=(r\cos t,r\sin t,0)$，求$\int_\alpha \kappa ds=$

s=s（t）$=\int_0^t\left|\dot{\alpha}(t)\right|dt=rt$，所以$\alpha(s)=(r\cos\dfrac{s}{r},r\sin\dfrac{s}{r},0)$

$t(s)=\dfrac{d\alpha}{ds}=(-\sin\dfrac{s}{r},\cos\dfrac{s}{r},0)$

$t'(s)=(-\dfrac{1}{r}\cos\dfrac{s}{r},-\dfrac{1}{r}\sin\dfrac{s}{r},0)=\dfrac{1}{r}n$，其中$n=(-\cos\dfrac{s}{r},-\sin\dfrac{s}{r},0)$，所以$\kappa=\dfrac{1}{r}$

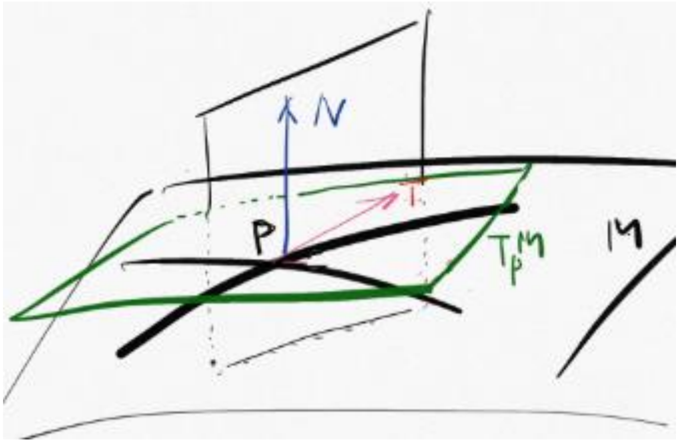
$\int_\alpha \kappa ds=\int_0^{2\pi}\dfrac{1}{r}ds=2\pi$，這是切線轉角定理。

$\dfrac{ds}{dt}=(\dot{X}\cdot\dot{X})^{\frac{1}{2}}$，$\dfrac{dt}{ds}=(\dot{X}\cdot\dot{X})^{-\frac{1}{2}}$，$\dfrac{d^2t}{ds^2}=\dfrac{d}{dt}(\dot{X}\cdot\dot{X})^{-\frac{1}{2}}\dfrac{dt}{ds}=...=-(\dot{X}\cdot\dot{X})^{-2}(\dot{X}\cdot\ddot{X})$

$X'=\dfrac{dX}{ds}=\dot{X}\dfrac{dt}{ds}=\dot{X}(\dot{X}\cdot\dot{X})^{-\frac{1}{2}}$

$X''=\ddot{X}(\dfrac{dt}{ds})^2+\dot{X}\dfrac{d^2t}{ds^2}=...=\dfrac{(\dot{X}\cdot\dot{X})\ddot{X}-(\dot{X}\cdot\ddot{X})\dot{X}}{(\dot{X}\cdot\dot{X})^2}$

（注意到$\left|\dot{X}\wedge\dot{X}\right|=0$，$(\dot{X}\cdot\dot{X})^{\frac{1}{2}}=\left|\dot{X}\right|$），Then $\kappa=\left|X'\wedge X''\right|=\dfrac{\left|\dot{X}\wedge\ddot{X}\right|}{\left|\dot{X}\right|^3}$



$\kappa = \frac{dT}{ds} = \kappa_n N + \kappa_g U$

U 落在切平面$T_p M$ 上,spanned by X_u, X_v

$\kappa_n = \frac{dT}{ds} \cdot N = -\frac{dN}{ds} \cdot T = \frac{II}{I}$ ---

$\rightarrow \begin{cases} H = \frac{1}{2}(\kappa_1 + \kappa_2) \\ K = \kappa_1 \kappa_2 = \frac{eg - f^2}{EG - F^2} \end{cases}$

取 lines of curvature 作參數座標時,F=f=0

$\kappa_1 = \frac{e}{E} ...(dv = 0), \kappa_2 = \frac{g}{G} ...(du = 0)$

$\kappa_g = \frac{dT}{ds} \cdot U$,$U = N \times T$ ---→測地線$< \frac{dT}{ds}, X_k >= 0$,得$u^{k''} + \Gamma_{ij}^k u^{i'} u^{j'} = 0$

κ_n的最大 最小值稱為主法曲率(principal normal curvature)

N 是曲面 M 上過 P 點的單位法向量
過 P 的垂直斷面所截的曲線 X
在 P 點的單位切向量為 T

$T = \frac{dX}{ds}$

κ_n是定義在 compact set 上的連續函數,所以其極值必存在

$\kappa_n = \frac{II}{I} = \frac{edu^2 + 2fdudv + gdv^2}{Edu^2 + 2Fdudv + Gdv^2} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2}$,其中$\lambda = \frac{dv}{du}$

$\frac{d\kappa_n}{d\lambda} = 0$時

$$(E + 2F\lambda + G\lambda^2)(2f + 2g\lambda) - (e + 2f\lambda + g\lambda^2)(2F + 2G\lambda) = 0$$

$$\frac{f + g\lambda}{F + G\lambda} = \frac{e + 2f\lambda + g\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{(e + f\lambda) + \lambda(f + g\lambda)}{(E + F\lambda) + \lambda(F + G\lambda)} = \frac{e + f\lambda}{E + F\lambda} \text{(合分比)}$$

$$(fG - gF)\lambda^2 + (eG - gE)\lambda + (eF - fE) = 0$$

$$(fG - gF)dv^2 + (eG - gE)dudv + (eF - fE)du^2 = 0 ...... \#$$

$$\begin{vmatrix} dv^2 & -dudv & dv^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$ ......此兩方向 $\frac{dv}{du}$, κ_n 有極值

由＃可證 principal direction 互相垂直,其積分曲線稱為 lines of curvature

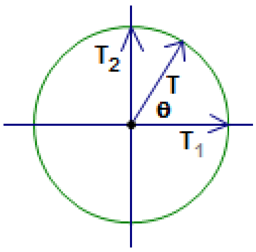
pf

$dX \perp \delta X \Leftrightarrow Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v = 0$

假設 $\lambda_1 = \frac{dv}{du}, \lambda_2 = \frac{\delta v}{\delta u}$,則 $E + F(\lambda_1 + \lambda_2) + G\lambda_1\lambda_2 = 0$

$$\begin{vmatrix} \lambda^2 & -\lambda & 1 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$ 的兩根為 λ_1, λ_2, $\lambda_1 + \lambda_2 = \frac{eG - Eg}{Fg - fG}, \lambda_1\lambda_2 = \frac{Ef - eF}{Fg - fG}$

則 $E + F(\lambda_1 + \lambda_2) + G\lambda_1\lambda_2 = 0$,得證



$T = \cos\theta T_1 + \sin\theta T_2$

則 $\kappa_n(\theta) = \cos^2\theta \cdot \kappa_1 + \sin^2\theta \cdot \kappa_2$......Euler 定理 1760

證 1

取 lines of curvature 作參數曲線 則 F=0

在 u-curve 上,du=0,在 v-curve 上,$\delta v = 0$

$$\begin{vmatrix} dv^2 & -dudv & dv^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$,所以 $\begin{cases} Ef - eF = 0 \\ gF - fG = 0 \end{cases}$,但 EG-F²>0,EG>o,所以 f=0

$$\kappa_n = \frac{edu^2 + gdv^2}{Edu^2 + Gdv^2} = e(\frac{du}{ds})^2 + g(\frac{dv}{ds})^2$$

$$\kappa_1 = \frac{e}{E}(\delta v = 0), \kappa_2 = \frac{g}{G}(du = 0)$$

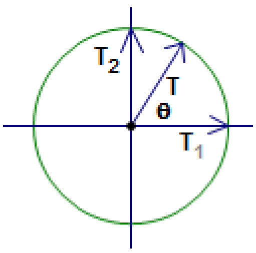
$$\cos\theta = \frac{dx \cdot \delta x}{|dx||\delta x|} = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{ds\sqrt{E\delta u^2 + 2F\delta u\delta v + G\delta v^2}}$$

$$= \frac{Edu\delta u}{ds\sqrt{E}\delta u} = \sqrt{E}\frac{du}{ds}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \ldots = \sqrt{G}\frac{dv}{ds}$$

$$\kappa_n = e(\frac{du}{ds})^2 + g(\frac{dv}{ds})^2 = \frac{e}{E}\cdot E(\frac{du}{ds})^2 + \frac{g}{G}\cdot G(\frac{dv}{ds})^2 = \kappa_1\cos^2\theta + \kappa_2\sin^2\theta$$

$\frac{1}{2\pi}\int_0^{2\pi}\kappa_n(T)d\theta = \frac{1}{2\pi}\int_0^{2\pi}(\kappa_1\cos^2\theta + \kappa_2\sin^2\theta)d\theta = \frac{\kappa_1+\kappa_2}{2}$ 這是均曲率的意義。



Euler 公式(證 2)

$A = -de_3$(即 do Carmo 的 dN_p)是 self-adjoint

即<Av,w>=<v,Aw>

當 $|T| = 1$ 時，$\kappa_n(T) = II(T) = < AT, T >$

T_1, T_2 是兩主法向量，則 $< AT_1, T_1 > = \kappa_1, < AT_2, T_2 > = \kappa_2$

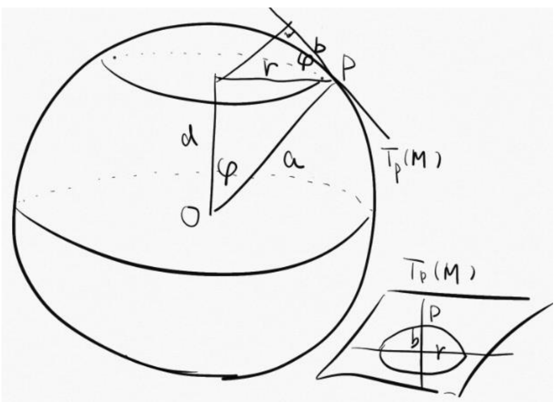
$$T = \cos\theta T_1 + \sin\theta T_2$$

$$\kappa_n = < AT, T > = < A(\cos\theta T_1 + \sin\theta T_2), \cos\theta T_1 + \sin\theta T_2 >$$

$$= < \cos\theta(AT_1) + \sin\theta(AT_2), \cos\theta T_1 + \sin\theta T_2 >$$

$$= \cos^2\theta < AT_1, T_1 > + \sin^2\theta < AT_2, T_2 >$$

$$= \kappa_1\cos^2\theta + \kappa_2\sin^2\theta$$



半徑=a 的球面上 小圓半徑 C=r

求在 P 點的曲率

小圓 C 在切平面的投影為一橢圓

$$\kappa_g(P) = \frac{b}{r^2} = \frac{r\cos\varphi}{r^2} = \frac{d}{ar}$$

[看 curvature01][Gauss-Bonett]

$\kappa = \frac{1}{r}\cos\varphi = \frac{d}{ar}$ 這是 Meusnier 定理

高斯曲率

X 是正交參數系，即 F=0，則 $K=-\frac{1}{2\sqrt{EG}}\left\{\left(\frac{E_v}{\sqrt{EG}}\right)_v+\left(\frac{G_u}{\sqrt{EG}}\right)_u\right\}$

Liouville

$$\left[\frac{Dw}{dt}\right]=\frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{dt}-E_v\frac{du}{dt}\right\}+\frac{d\varphi}{dt}$$

所以 $\kappa_g=\frac{1}{2\sqrt{EG}}\left\{G_u\frac{dv}{ds}-E_v\frac{du}{ds}\right\}+\frac{d\varphi}{ds}$

$(\kappa_g)_1$ 是 u-curve 的 geodesic，v=const 則

(1) $\frac{dv}{ds}=0$ (2) $ds^2=Edu^2$，$\therefore\frac{du}{ds}=\frac{1}{\sqrt{E}}$

$$(\kappa_g)_1=\frac{1}{2\sqrt{EG}}\times E_v\times\frac{1}{\sqrt{E}}=-\frac{E_v}{2E\sqrt{G}}$$

同理 $(\kappa_g)_2=\frac{G_u}{2G\sqrt{E}}$

$$\kappa_g=\frac{1}{2\sqrt{EG}}\left\{-E_v\frac{du}{ds}+G_u\frac{dv}{ds}\right\}+\frac{d\varphi}{ds}\text{......(*)}$$

($e_1=\frac{X_u}{\sqrt{E}},e_2=\frac{X_v}{\sqrt{G}}$ ，$\alpha'=X_u\frac{du}{ds}+X_v\frac{dv}{ds}=e_1\cos\varphi+e_2\sin\varphi$

$\sqrt{E}\frac{du}{ds}=\cos\varphi$ ，$\sqrt{G}\frac{dv}{ds}=\sin\varphi$ 所以 $\kappa_g=(\kappa_g)_1\cos\varphi+(\kappa_g)_2\sin\varphi+\frac{d\varphi}{ds}$)

其中 $(\kappa_g)_1\cos\varphi=\frac{-E_v}{2E\sqrt{G}}\cdot\sqrt{E}\frac{du}{ds}=\frac{-E_v}{2\sqrt{EG}}\frac{du}{ds}$

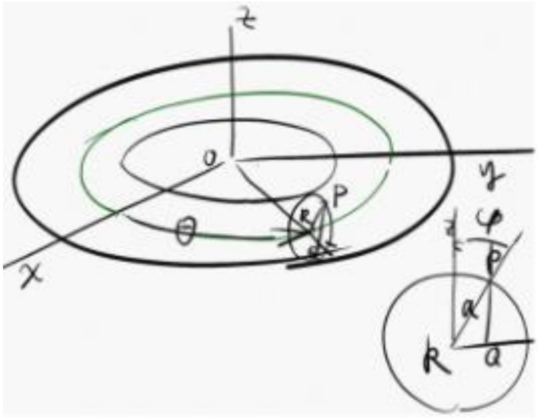
$$(\kappa_g)_2\sin\varphi=...=\frac{G_u}{2\sqrt{EG}}\frac{dv}{ds}$$

$$\int_C(\kappa_g)_1\cos\varphi ds=\int_C\left(\frac{-E_v}{2\sqrt{EG}}\right)du=\iint_A\frac{\partial}{\partial v}\left(\frac{E_v}{2\sqrt{EG}}\right)dudv$$

同理 $\int_C(\kappa_g)_2\sin\varphi ds=\int_C\frac{G_u}{2\sqrt{EG}}dv=-\iint_A\frac{\partial}{\partial u}\left(\frac{G_u}{2\sqrt{EG}}\right)dudv$

所以 最後還是這個問題

$$\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right)=\frac{1}{\sqrt{EG}}\left(\frac{E_v}{\sqrt{EG}}\right)$$



輪胎面(torus)的高斯曲率

管心曲線為 $y(\theta)=[b\cos\theta, b\sin\theta, 0]$

$$\overline{OQ}=\overline{OR}+\overline{RQ}=b+a\sin\varphi$$

$$X(\theta,\varphi)=[(b+a\sin\varphi)\cos\theta,(b+a\sin\varphi)\sin\theta,a\cos\varphi]$$

$$X_\theta=[-(b+a\sin\varphi)\sin\theta,(b+a\sin\varphi)\cos\theta,0]$$

$$X_\varphi=[a\cos\varphi\cos\theta,a\cos\varphi\sin\theta,-a\sin\varphi]$$

$$E=(b+a\sin\varphi)^2, F=0, G=a^2$$

$$X_\theta\times X_\varphi=a(b+a\sin\varphi)[-\cos\theta\sin\varphi,-\sin\theta\sin\varphi,-\cos\varphi]$$

所以 $N=[-\cos\theta\sin\varphi,-\sin\theta\sin\varphi,-\cos\varphi]$

$$X_{\theta\theta}=\ldots, X_{\theta\varphi}=\ldots, X_{\varphi\varphi}=\ldots$$

$$e=X_{\theta\theta}\cdot N=(b+a\sin\varphi)\sin\varphi$$

$$f=X_{\theta\varphi}\cdot N=0$$

$$g=X_{\varphi\varphi}\cdot N=a^2$$

$$K=\frac{eg-f^2}{EG-F^2}=\frac{a(b+a\sin\varphi)\sin\varphi}{a^2(b+a\sin\varphi)^2}=\frac{\sin\varphi}{a(b+a\sin\varphi)}$$