

§ metric

A Riemann manifold is a pair (M, g) consisting of a smooth manifold M and a metric g on the tangent bundle.

Two Riemann manifolds (M_1, g_1) and (M_2, g_2) are said to be isometric if there exists a

diffeomorphism $\phi: M_1 \rightarrow M_2$ such that $\phi^* g_2 = g_1$

流形 M 上存在處處可微、恆正、二階對稱協變張量場 G ，稱為度規張量場。

$$g_{ij} = g_{ji}, \det(g_{ij}) \neq 0$$

恆正改為非奇異(nonsingular)，則相應的流形 M 稱為廣義黎曼流形。

在 4-dim 的 Spacetime 上的流形是廣義的黎曼流形。

定義 $ds^2 = g_{ij} dx^i dx^j$ ，則弧長 $\Delta s = \int (g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt})^{\frac{1}{2}} dt$

利用 metric 可在流形 M 上每一點 p 的切空間 $T_p M$ 給出兩切向量的內積，定義

向量的長度，兩向量的夾角。

利用聯絡結構可定義張量場的協變微分與平行運輸(parallel transport)，向量場依此聯絡平行運輸時保持向量長度不變及向量間夾角不變，即度規張量場 G 的協變微分為零， $\nabla G = 0$

取局部座標時可表為 $\nabla g = 0$

§ induced metric

$\varphi: M \rightarrow N$ is an immersion

(N, g) is a Riemannian manifold, then $\varphi^* g$ is a Riemannian metric in M induced by φ

Example

$(R^3, g), g = (dx)^2 + (dy)^2 + (dz)^2$, $\varphi: S^2 \rightarrow R^3$ 求 $\varphi^* g$

$$\varphi(x, y) = (x, y, \sqrt{1-x^2-y^2})$$

$$\frac{\partial \varphi}{\partial x} = (1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}), \quad \frac{\partial \varphi}{\partial y} = (0, 1, \frac{-y}{\sqrt{1-x^2-y^2}})$$

$$g_{11} = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial x} \rangle = \frac{1-y^2}{1-x^2-y^2}, \quad g_{12} = g_{21} = \langle \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \rangle = \frac{2xy}{1-x^2-y^2},$$

$$g_{22} = \langle \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial y} \rangle = \frac{1-x^2}{1-x^2-y^2}$$

$$\text{So } \phi^*g = \frac{1-y^2}{1-x^2-y^2}(dx)^2 + \frac{2xy}{1-x^2-y^2}dxdy + \frac{1-x^2}{1-x^2-y^2}(dy)^2$$

Then ϕ is called isometric embedding(同維度時才是 isometry(等距同構))

§ (M, g) (N, h) are Riemannian manifolds , $f: M \rightarrow N$

If $f^*h = g$ then f is called an isometry .

An isometry is a metric-preserving diffeomorphism .

Example

$C: \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a cylinder

$\phi(x, y, z) = (x, -y, -z): C \rightarrow C$ then

(1) ϕ is an isometry

(2) $\phi(p) = p$, $p=?$

(2) $P(1,0,0)$ $(-1,0,0)$

Example

$\phi: S^2 \rightarrow \mathbb{R}^3$, $\phi: (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3$

Then $\phi^*g = d\theta^2 + \sin^2 \theta d\varphi^2$ is the induced metric on S^2

$\phi(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$

$$\frac{\partial \phi}{\partial \theta} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)$$

$$\frac{\partial \phi}{\partial \varphi} = (-\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0)$$

$$g_{11} = 1, g_{12} = g_{21} = 0, g_{22} = \sin^2 \theta$$

Then $\phi^*g = d\theta^2 + \sin^2 \theta d\varphi^2$

或者 on S^2 $x = \sin \theta \cos \varphi, y = \sin \theta \sin \varphi, z = \cos \theta$

$$dx = \cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi$$

$$dy = \cos \theta \sin \varphi d\theta + \sin \theta \cos \varphi d\varphi$$

$$dz = -\sin \theta d\theta$$

Then $dx^2 + dy^2 + dz^2 = \dots = d\theta^2 + \sin^2 \theta d\varphi^2$

$(S^2, h) \xrightarrow{\phi} (\mathbb{R}^3, g)$, $g = dx^2 + dy^2 + dz^2, h = d\theta^2 + \sin^2 \theta d\varphi^2$

$$\phi^*g = d\theta^2 + \sin^2 \theta d\varphi^2 = h$$

Exercise

Let $H^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. At each point $p = (x, y) \in H^2$, define

$$\langle \cdot, \cdot \rangle_{H^2} : T_p H^2 \times T_p H^2 \rightarrow \mathbb{R} \text{ by } \langle u, v \rangle_{H^2} = \frac{1}{y^2} \langle u, v \rangle$$

Where $\langle \cdot, \cdot \rangle$ is the usual Euclidean inner product.

Show that $\langle \cdot, \cdot \rangle_{H^2}$ is a Riemannian metric on H^2 .

§ hyperbolic geometry

Let $H = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ denote the upper half plane.

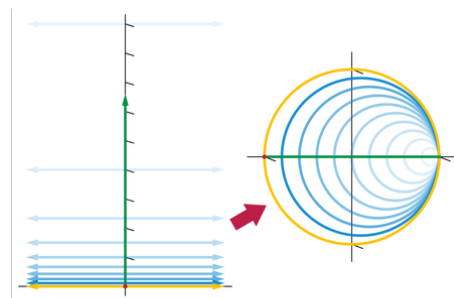
Endowed with the metric $h = \frac{1}{4v^2} (du^2 + dv^2)$

The Poincaré model of the hyperbolic plane is the Riemann manifold (D, g) where D is the unit open disk in the plane \mathbb{R}^2 and the metric g is given by

$$g = \frac{1}{1-x^2-y^2} (dx^2 + dy^2)$$

Show that the Cayley transform $z = x + iy \rightarrow w = -i \frac{z+i}{z-i} = u + iv$ establishes an

isometry $(D, g) \cong (H, h)$



Cayley transform of upper complex half-plane to unit disk

$\varphi(z) = \frac{z-i}{z+i} : H \rightarrow D$ is the Cayley transform,

which maps $\{\infty, 1, -1, i, 0\} \rightarrow \{1, -i, i, 0, -1\}$

$$z = x \in \mathbb{R}, \text{ then } u + iv = \frac{x-i}{x+i} \Rightarrow u^2 + v^2 = 1$$

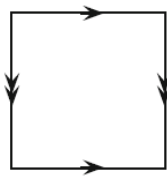
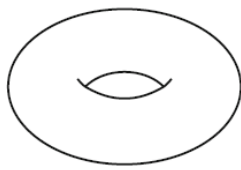
i.e. x -axis $\xrightarrow{\varphi}$ unit circle

Note about Möbius transform

$$\text{If } f(z) = \frac{az+b}{cz+d} \text{ then } f^{-1}(z) = \frac{-dz+b}{cz-a};$$

$$\text{If } f(z) = -i \times \frac{z+i}{z-i} \text{ then } f^{-1}(z) = \frac{iz+1}{z+i} = i \times \frac{z-i}{z+i}$$

§ A torus T^2 in R^3 inherits the Euclidean metric from R^3 .



A torus is also $T^2 \cong R^2 / Z^2$ as a quotient space , it inherits a Riemannian metric from R^2 .

Fig. 1.2. Two Riemannian metrics on the torus.

With these two Riemannian metrics , the torus becomes two distinct Riemannian manifolds .

Show that there is no isometry between these two Riemannian manifolds with the same underlying torus .

§ Metric connection

We say that a connection ∇ on a Riemannian bundle E is compatible with the metric if for all $X \in \chi(M)$ and $s, t \in \Gamma(E)$

$$X \langle s, t \rangle = \langle \nabla_X s, t \rangle + \langle s, \nabla_X t \rangle$$

Then ∇ is called metric connection .

$$\text{「相容 } X \cdot \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

對稱 $\nabla_X Y - \nabla_Y X = [X, Y]$ (called torsion free)

When ∇ compatible with metric and torsion free then ∇ is called Levi-Civita connection (Riemannian connection) . 」