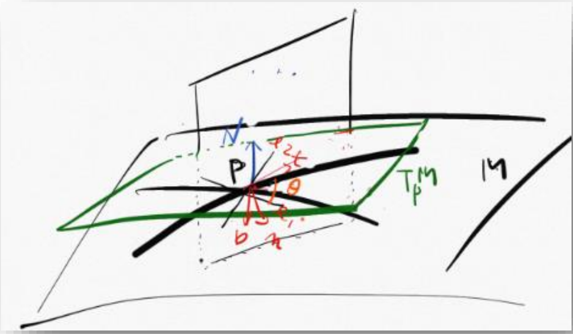
§ Structure Equation



曲面論的活動標架法

$$\vec{\xi}_1 = \frac{X_u}{|X_u|}，\vec{\xi}_3 = \frac{X_u \times X_v}{|X_u \times X_v|}，$$

$$\vec{\xi}_2 = \vec{\xi}_3 \times \vec{\xi}_1$$

$$A = (\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3) = \begin{pmatrix} \xi_1^1 & \xi_2^1 & \xi_3^1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 \\ \xi_1^3 & \xi_2^3 & \xi_3^3 \end{pmatrix}$$

$$X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}, \hat{\theta} = \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix}$$

$$\omega = A^t dA = \begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} = \left(\omega_j^i\right) = (<\overline{\xi_i}, d\overline{\xi_j}>)$$ ，稱為 matrix of connection form

其中$\eta^0 = <\vec{\xi}_2, d\vec{\xi}_1>, \eta^1 = <\vec{\xi}_3, d\vec{\xi}_1>, \eta^2 = <\vec{\xi}_3, d\vec{\xi}_2>$

$\hat{\theta} = A^t dX$ ，則 $dX = A\hat{\theta}, dA = A\omega$

(then $\theta^i = \overline{\xi_i} \cdot dX$)

由 d(d X)=0 推出 $d\hat{\theta} = -\omega \wedge \hat{\theta} \Rightarrow d\theta^1 = \eta^0 \wedge \theta^2, d\theta^2 = -\eta^0 \wedge \theta^1, \eta^1 \wedge \theta^1 + \eta^2 \wedge \theta^2 = 0$

由 d(d A)=0 推出 $d\omega = -\omega \wedge \omega \Rightarrow \begin{cases} d\eta^0 = -\eta^1 \wedge \eta^2 \\ d\eta^1 = -\eta^2 \wedge \eta^0 \\ d\eta^2 = -\eta^0 \wedge \eta^1 \end{cases}$

$0 = d(dX) = d(A\hat{\theta}) = dA \wedge \hat{\theta} + Ad\hat{\theta}$

$= (A\omega) \wedge \hat{\theta} + Ad\hat{\theta}$　　　　所以 $d\hat{\theta} = -\omega \wedge \hat{\theta}$

$= A(\omega \wedge \hat{\theta} + d\hat{\theta})$

同理 由 d(d A)=0 推出 $d\omega = -\omega \wedge \omega$

當 X=X(u,v)

$dX = X_u du + X_v dv$

$\theta^1 = <\overrightarrow{\xi_1}, dX> = <\overrightarrow{\xi_1}, X_u> du + <\overrightarrow{\xi_1}, X_v> dv$

$\theta^2 = <\overrightarrow{\xi_2}, dX> = <\overrightarrow{\xi_2}, X_u> du + <\overrightarrow{\xi_2}, X_v> dv$

$\theta^3 = <\overrightarrow{\xi_3}, dX> = 0$

$$d\hat{\theta} = -\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^1 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}\begin{pmatrix} \theta^1 \\ \theta^2 \\ 0 \end{pmatrix}$$

$d\theta^1 = \eta^0 \wedge \theta^2$

$d\theta^2 = -\eta^0 \wedge \theta^1$

$d\theta^3 = -\eta^1 \wedge \theta^1 - \eta^2 \wedge \theta^2 = 0$，所以$\eta^1 \wedge \theta^1 + \eta^2 \wedge \theta^2 = 0$

$$d\omega = -\omega \wedge \omega = -\begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^0 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\eta^0 & -\eta^1 \\ \eta^1 & 0 & -\eta^2 \\ \eta^1 & \eta^2 & 0 \end{pmatrix}$$ ，得

$d\eta^0 = -\eta^1 \wedge \eta^2$ 是 Gauss equation

$\begin{cases} d\eta^1 = -\eta^2 \wedge \eta^0 \\ d\eta^2 = -\eta^0 \wedge \eta^1 \end{cases}$ 是 Codazzi-Mainardi equation

定義	identities	structure equation
$\hat{\theta} = A^t dX$	$A^T A = I$	$d\hat{\theta} = -\omega \wedge \hat{\theta}$
$\omega = A^t dA$	$\omega + \omega^T = 0$	$d\omega = -\omega \wedge \omega$

$dX = X_u du + X_v dv = \theta^1 \vec{\xi_1} + \theta^2 \vec{\xi_2}$ 推出 Area form $\theta^1 \wedge \theta^2 = |X_u \times X_v| dudv$

定義 $\eta^1 \wedge \eta^2 = K(\theta^1 \wedge \theta^2)$

$-\theta^2 \wedge \eta^1 + \theta^1 \wedge \eta^2 = 2H(\theta^1 \wedge \theta^2)$

因為$\theta^1 \wedge \theta^2 \neq 0$，$\theta^1, \theta^2$構成 M 上每一點 1-form 的基底

設 $\eta^1 = a\theta^1 + b\theta^2, \eta^2 = c\theta^1 + d\theta^2$

因為 $\eta^1 \wedge \theta^1 + \eta^2 \wedge \theta^2 = 0$ 所以 b=c

Rewrite let $\eta^1 = a\theta^1 + c\theta^2, \eta^2 = c\theta^1 + b\theta^2$

$$\begin{pmatrix}\eta^1\\ \eta^2\end{pmatrix}=\begin{pmatrix}a & c\\ c & b\end{pmatrix}\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}=M\begin{pmatrix}\theta^1\\ \theta^2\end{pmatrix}$$

$K=\det M=\kappa_1\kappa_2=ab-c^2$ 其中 κ_1,κ_2 是 M 的 eigenvalue

因為 determinant 與 trace 對角化的不變量

$H=trM=\dfrac{\kappa_1+\kappa_2}{2}$

古典微分幾何

$\dfrac{dT}{ds}=\kappa_n N+\kappa_g Y$ ，$\kappa_n=\dfrac{II}{I}$ 其中 $II=-dX\cdot dN=edu^2+2fdudv+gdv^2$

$e=X_{uu}\cdot N, f=X_{uv}\cdot N, g=X_{vv}\cdot N$

$H=\dfrac{Eg-2Ff+Ge}{2(EG-F^2)}$ ，$K=\dfrac{eg-f^2}{EG-F^2}$

R^n 中的結構方程

設 $dX=\omega^A e_A$ ，$de_A=\omega_A^B e_B$ ，則結構方程為

(1) $d\omega^A=\omega^B\wedge\omega_B^A$

(2) $d\omega_A^B=\omega_A^C\wedge\omega_C^B$

Cartan structure quation

A field of frames $\{X_1,X_2,...,X_n\}, X_i=\dfrac{\partial}{\partial x^i}$

A field of dual frames $\{\omega^1,\omega^2,...,\omega^n\}$

Levi-Civita connection $\nabla_X Y$ ，$\nabla_{X_i}=\sum_k \Gamma_{ij}^k X_k$

$X=\sum_i a^i X_i, Y=\sum_i b^i X_i$ then $\nabla_X Y=\sum_j (X\cdot b^i+\sum_{i=1} b^i\omega_i^j(X))X_j$

其中 $\omega_j^k=\sum_i \Gamma_{ij}^k\omega^i$ or $\Gamma_{ij}^k=\omega_j^k(X_i)$

Riemannian curvature tensor $R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$

定義 $\omega_j^k=\sum_i \Gamma_{ij}^k\omega^i$

定理 (Cartan) 證明過程在[黎曼幾何簡介 p.133]

1. $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$

2. $dg_{ij} = \sum_k (g_{kj}\omega_i^k + g_{ki}\omega_j^k)$ if $g_{ij} = \delta_{ij}$ then $\omega_i^j + \omega_j^i = 0$

3. $\Omega_k^l(X,Y) = \omega^l(R(X,Y),X_k)$ then $\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j$

(1)(2)(3)稱為 Cartan 結構方程式

定理

If M is a 2-dim manifold ，then for an orthonormal frame $\Omega_1^2 = -K\omega^1 \wedge \omega^2$

Where K is the Gauss curvature of M

p.139 定義 geodesic curvature of curve c(s) $\kappa_g(s) := \omega_1^2(E_1(s))$

例題

1. $S^2 = (a\cos u \sin v, a\sin u \sin v, a\cos v)$

計算 Gauss curvature K

$X_u =$

$X_v =$

$\vec{\xi}_1 =$

$\vec{\xi}_2 =$

$\vec{\xi}_3 =$

$\theta^1 = a\sin v du, \theta^2 = adv$

$\eta^1 = \sin v du, \eta^2 = dv$

$K = \frac{1}{a^2}, H = \frac{1}{a}$

2. Hyperbolic plane $g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$

計算 Gauss Curvature K

$X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y}$ Then $< X_1, X_1 >= \frac{1}{y^2}, < X_1, X_2 >= 0, < X_2, X_2 >= \frac{1}{y^2}$

取 $E_1 = yX_1, E_2 = yX_2$ form an orthonormal frames

則 $\omega^1 = \frac{1}{y}dx, \omega^2 = \frac{1}{y}dy$

$d\omega^1 = (-\frac{1}{y^2})dy \wedge dx = \frac{1}{y^2}dx \wedge dy = \omega^1 \wedge \omega^2$

$d\omega^2 = 0$

由 $d\omega^i = \sum_j \omega^j \wedge \omega^i_j$

$d\omega^1 = \omega^1 \wedge \omega^1_1 + \omega^2 \wedge \omega^1_2 = \omega^2 \wedge \omega^1_2 = \omega^2_1 \wedge \omega^2$

$d\omega^2 = \omega^1 \wedge \omega^2_1 + \omega^2 \wedge \omega^2_2 = \omega^1 \wedge \omega^2_1 = 0$

Let $\omega^2_1 = a\omega^1 + b\omega^2$ ， $\omega^1 \wedge \omega^2_1 = 0$ 所以 b=0

$\omega^2_1 \wedge \omega^2 = d\omega^1 = \omega^1 \wedge \omega^2 = a\omega^1 \wedge \omega^2$ 所以 a=1

$d\omega^2_1 = d\omega^1 = \omega^1 \wedge \omega^2 = -K\omega^1 \wedge \omega^2$ 所以 K=-1

3. Helicoid (螺旋曲面)

$X(u,v) = (u\cos v, u\sin v, v)$, $p = \sqrt{1+u^2}$

$X_u =$

$X_v =$

$\vec{\xi_1} =$

$\vec{\xi_3} =$

$\vec{\xi_2} = \vec{\xi_3} \times \vec{\xi_1} =$

$\theta^1 = <\vec{\xi_1}, X_u> du + <\vec{\xi_1}, X_v> dv = du$ ， $\theta^2 = <\vec{\xi_2}, X_u> du + <\vec{\xi_2}, X_v> dv = pdv$

$\eta^1 = <\vec{\xi_3}, d\vec{\xi_1}> = -\frac{1}{p}dv$ ， $\eta^2 = <\vec{\xi_3}, d\vec{\xi_2}> = -\frac{1}{p^2}du$

Then H=0， $K = -\frac{1}{p^4} = -\frac{1}{(1+u^2)^2}$

4. S^2 $X(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$g = A^2(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta d\varphi \otimes d\varphi$ on $M = I \times S^2$

計算 Ricci 張量

$R_{iji}^{\ \ j} = \Omega_i^j(E_i, E_j), R_{ij} = R_{kij}^k$

$X_1 = \frac{\partial}{\partial r}, X_2 = \frac{\partial}{\partial \theta}, X_3 = \frac{\partial}{\partial \varphi}$

取 $E_1 = \frac{1}{A(r)} X_1, E_2 = \frac{1}{r} X_2, E_3 = \frac{1}{r\sin\theta} d\varphi$

Then $\omega^1 = A(r)dr, \omega^2 = rd\theta, \omega^3 = r\sin\theta d\varphi$

先由 $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$ 計算 $d\omega^1 =$ $d\omega^2 =$ $d\omega^3 =$

$\omega_1^2 =$

再由 $\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j$ 計算 $\Omega_1^2 =$ $\Omega_1^3 =$ $\Omega_2^3 =$

$R_{121}^2 =$

$R_{131}^3 =$

$R_{232}^3 =$ 最後

$R_{11} = R_{111}^1 + R_{211}^2 + R_{311}^3 = \ldots = \frac{-A'}{rA^3}$

$R_{22} = R_{122}^1 + R_{222}^2 + R_{322}^3 = \ldots = \frac{A'}{rA^3} - \frac{1}{r^2}(\frac{1}{A^2} - 1)$

$R_{33} = R_{133}^1 + R_{233}^2 + R_{333}^3 = \ldots = \frac{A'}{rA^3} - \frac{1}{r^2}(\frac{1}{A^2} - 1)$

5. …

[大域微分幾何]p.67 球面 S^n 的曲率

截曲率 $R_{ijij} = \frac{1}{r^2}, \forall i, j, i \neq j$ 當 n=2 時 $R_{1212} = \frac{1}{r^2}$ 就是 Gauss 曲率。

p.63 § 3 計算 Clifford 環面的曲率

[大域微分幾何] p.346 結構方程在曲面論的應用

Exercise 2.8

(1) Let $\{X_1, \ldots, X_n\}$ be a field of frames on an open set V of a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ with Levi–Civita connection ∇. The associated **structure functions** C_{ij}^k are defined by

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k.$$

Show that:

(a) $C_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i$;

(b) $\Gamma_{jk}^i = \frac{1}{2} \sum_{l=1}^n g^{il} \left(X_j \cdot g_{kl} + X_k \cdot g_{jl} - X_l \cdot g_{jk} \right)$
$+ \frac{1}{2} C_{jk}^i - \frac{1}{2} \sum_{l,m=1}^n g^{il} \left(g_{jm} C_{kl}^m + g_{km} C_{jl}^m \right)$;

(c) $d\omega^i + \frac{1}{2} \sum_{j,k=1}^n C_{jk}^i \omega^j \wedge \omega^k = 0$, where $\{\omega^1, \ldots, \omega^n\}$ is the field of dual coframes.

(2) Let $\{X_1, \ldots, X_n\}$ be a field of frames on an open set V of a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$. Show that a connection ∇ on M is compatible with the metric on V if and only if

$$X_k \cdot \langle X_i, X_j \rangle = \langle \nabla_{X_k} X_i, X_j \rangle + \langle X_i, \nabla_{X_k} X_j \rangle$$

for all i, j, k.

(3) Compute the Gauss curvature of:

(a) the sphere S^2 with the standard metric;

(b) the hyperbolic plane, i.e. the upper half-plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the metric

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

[cf. Exercise 3.3(5) of Chap. 3].

(4) Determine all surfaces of revolution with constant Gauss curvature.

(5) Let M be the image of the parameterization $\varphi : (0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u \cos v, u \sin v, v),$$

and let N be the image of the parameterization $\psi : (0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\psi(u, v) = (u \cos v, u \sin v, \log u).$$

Consider in both M and N the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$. Show that the map $f : M \to N$ defined by

$$f(\varphi(u, v)) = \psi(u, v)$$

preserves the Gauss curvature but is not a local isometry.

(6) Consider the metric

$$g = A^2(r)dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta\, d\varphi \otimes d\varphi$$

on $M = I \times S^2$, where r is a local coordinate on $I \subset \mathbb{R}$ and (θ, φ) are spherical local coordinates on S^2.

(a) Compute the Ricci tensor and the scalar curvature of this metric.
(b) What happens when $A(r) = (1 - r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to S^3)?
(c) And when $A(r) = (1 + r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to the **hyperbolic** 3-**space**)?
(d) For which functions $A(r)$ is the scalar curvature constant?

(7) Let M be an oriented Riemannian 2-manifold and let p be a point in M. Let D be a neighborhood of p in M homeomorphic to a disc, with a smooth boundary ∂D. Consider a point $q \in \partial D$ and a unit vector $X_q \in T_q M$. Let X be the parallel transport of X_q along ∂D in the positive direction. When X returns to q it makes an angle $\Delta\theta$ with the initial vector X_q. Using fields of positively oriented orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ such that $F_1 = X$, show that

$$\Delta\theta = \int_D K.$$

Conclude that the Gauss curvature of M at p satisfies

$$K(p) = \lim_{D \to p} \frac{\Delta\theta}{\operatorname{vol}(D)}.$$

(8) Compute the geodesic curvature of a positively oriented circle on:

(a) $\mathbb{R}^2$ with the Euclidean metric and the usual orientation;
(b) S^2 with the usual metric and orientation.

(9) Let c be a smooth curve on an oriented 2-manifold M as in the definition of geodesic curvature. Let X be a vector field parallel along c and let θ be the angle between X and $\dot{c}(s)$ along c in the given orientation. Show that the geodesic curvature of c, k_g, is equal to $\frac{d\theta}{ds}$. (**Hint:** Consider two fields of orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ positively oriented such that $E_1 = \frac{X}{\|X\|}$ and $F_1 = \dot{c}$).