

§ Stokes 定理  $\int_D \partial w = \int_{\partial D} w$

梯度 旋度 散度

Poincare lemma  $d(dw) = 0$

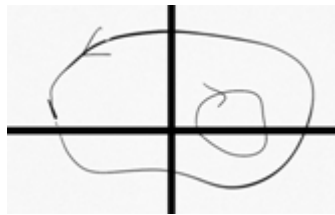
一.

$f$  是可微分函數 0-form

$w = f, D = [a, b], dw = df = f' dx$

則  $\int_D dw = \int_D f'(x) dx = \int_{\partial D} f = f(b) - f(a)$  這是微積分基本定理

二.



$w = P dx + Q dy$  1-form

$$dw = \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

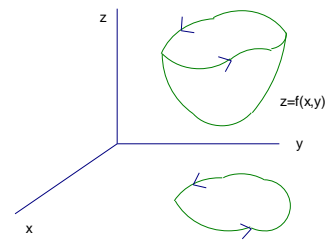
則  $\int_{\partial D} P dx + Q dy = \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$  ...Green 定理

三.

$D$  是  $R^3$  中的有向曲面,  $\partial D$  是有向閉曲線

$w = P dx + Q dy + R dz$  ...1-form

$$dw = \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}, \text{ 則}$$



$\oint_S \vec{E} \cdot \vec{n} ds = \iiint_V (\text{curl } \vec{E}) \cdot \vec{n} dV$  ...Stokes 定理

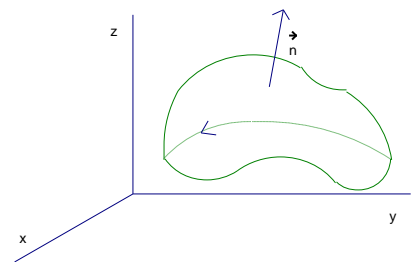
四.

$D$  是  $R^3$  中的有界區域

$w = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy$  ...2-form

則  $dw = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$

$\iiint_V \vec{E} \cdot \vec{n} dV = \iiint_V \text{div } \vec{E} dV$  ...Gauss 定理 (散度定理)



[誰怕向量微積分 Div, Grad, Curl, and All That H.M.Schey]

在靜電場中

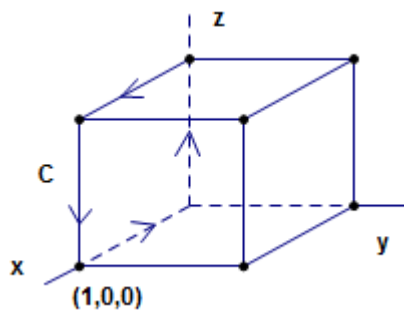
高斯定律  $\iiint_S \vec{E} \cdot \vec{n} dS = 4\pi q$ ,  $\iint_S \vec{F} \cdot \vec{n} dS$  稱為通量 (Flux)

$$\operatorname{div} E = \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S E \cdot \vec{n} dS = 4\pi\rho \text{ (單位體積的通量)} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

可以推出連續方程

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0, \text{ 其中 } J = \rho V$$

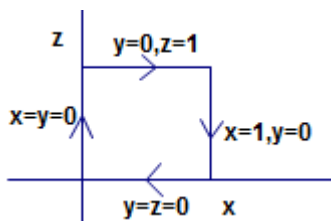
例1. Stokes 定理習作



$$F = [z^2, -y^2, 0]$$

$$\text{驗證 } \oint_C F \cdot tds = \iint_S \vec{n} \cdot \operatorname{curl} F dS$$

$$\operatorname{curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -y^2 & 0 \end{vmatrix} = [0, 2z, 0]$$



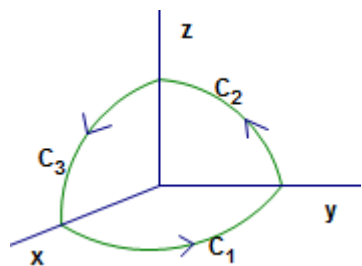
$$\oint_C F \cdot tds = \oint_C (F_x dx + F_y dy + F_z dz)$$

$$= \oint_C (z^2 dx - y^2 dy) = \int_0^1 dx = 1$$

覆蓋面  $S_1 \sim S_5$  只有  $S_5$ ,  $\vec{n} = [0, 1, 0]$ , 積分值不為 0

$$\iint_{S_5} \vec{n} \cdot \operatorname{curl} F dS = \int_0^1 \int_0^1 2z dx dz = 1$$

例2.



$$F = [y, z, x]$$

$$S: x^2 + y^2 + z^2 = 1$$

$$0 \leq x, y, z \leq 1$$

C 由  $C_1, C_2, C_3$  構成

$$\oint_C F \cdot tds = \oint_C (y dx + z dy + x dz)$$

$$\operatorname{curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = [-1, -1, -1], \text{ S: } x^2 + y^2 + z^2 = 1, \vec{n} = [x, y, z]$$

$$z = f(x, y) = \sqrt{1-x^2-y^2}, \quad \frac{\partial f}{\partial x} = -\frac{x}{z}, \quad \frac{\partial f}{\partial y} = -\frac{y}{z}$$

$$\iint_S \vec{n} \cdot \text{curl} F dS = \iint_R -(x+y+\sqrt{1-x^2-y^2}) \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} dx dy$$

$$\text{let } x = r \cos \theta, y = r \sin \theta, 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 -(r \cos \theta + r \sin \theta + \sqrt{1-r^2}) \frac{1}{\sqrt{1-r^2}} r dr d\theta$$

$$\oint_C (y dx + z dy + x dz) = \frac{3\pi}{4}$$

$$\int \frac{x^2 dx}{\sqrt{1-x^2}} = -\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x + C$$

$$\int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr = -\frac{r}{2} \sqrt{1-r^2} + \frac{1}{2} \sin^{-1} r \Big|_0^1 = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \frac{\pi}{4} (\cos \theta + \sin \theta) d\theta = \frac{\pi}{4} (\sin \theta - \cos \theta) \Big|_0^{\frac{\pi}{2}} = \frac{2\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 r dr d\theta = \frac{\pi}{4}$$

例3. 求  $\iint_S z^2 dS$ , 其中  $S$  是在第一卦限的  $\frac{1}{8}$  球面  $\frac{\pi}{6}$

$$X(x, y) = (x, y, \sqrt{1-x^2-y^2})$$

$$X_x = [1, 0, \frac{-x}{\sqrt{1-x^2-y^2}}]$$

$$X_y = [0, 1, \frac{-y}{\sqrt{1-x^2-y^2}}]$$

$$E = \frac{1-y^2}{1-x^2-y^2}, F = \frac{xy}{1-x^2-y^2}, G = \frac{1-x^2}{1-x^2-y^2}$$

$$dS = \sqrt{EG - F^2} dx dy = \frac{1}{z} dx dy$$

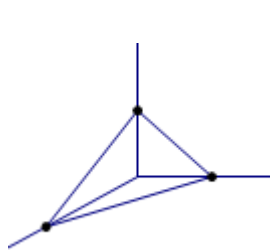
$$x = r \cos \theta, y = r \sin \theta$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\iint_S z^2 dS = \iint_R \sqrt{1-x^2-y^2} dx dy = \int_0^{\frac{\pi}{2}} \int_0^1 r \sqrt{1-r^2} dr d\theta = \frac{1}{3} \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{6}$$

例4. 求  $\iint_S F \cdot \bar{n} dS$ ，其中 S 是  $x+2y+2z=1$  與第一卦限截出之曲面

$$F = (z, -y, x) \quad \frac{1}{2}$$



$$X = [x, y, 1 - \frac{1}{2}x - y]$$

$$X_x = [1, 0, -\frac{1}{2}], X_y = [0, 1, -1]$$

$$E = \frac{5}{4}, F = \frac{1}{2}, G = 2, dS = \sqrt{EG - F^2} dx dy = \frac{3}{2} dx dy$$

$$\bar{n} = [\frac{1}{3}, \frac{2}{3}, \frac{2}{3}], F \cdot \bar{n} = \frac{1}{3}z - \frac{2}{3}y + \frac{2}{3}x = \frac{1}{3} + \frac{1}{2}x - y$$

$$\iint_S F \cdot \bar{n} dS = \iint_R \frac{3}{2} (\frac{1}{3} + \frac{1}{2}x - y) dx dy$$

$$\int_0^{2-2y} (\frac{1}{2} + \frac{3}{4}x - \frac{3}{2}y) dx = \dots = \frac{9}{2}y^2 - 7y + \frac{5}{2}$$

$$\iint_R \frac{3}{2} (\frac{1}{3} + \frac{1}{2}x - y) dx dy = \int_0^1 (\frac{9}{2}y^2 - 7y + \frac{5}{2}) dy = \frac{1}{2}$$

在  $R^3$  中

$$\nabla = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]$$

$$\text{grad} f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}]$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$\text{curl}F = \nabla \times F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\omega = Pdx + Qdy + Rdz \quad \text{則} \quad d\omega = \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\text{div}F = \nabla \cdot F = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}$$

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \quad \text{則} \quad d\omega = \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

則  $\text{curl}(\text{grad}f) = 0, \text{div}(\text{curl}F) = 0$ , 即  $\mathbb{R}^3$  中  $d(d\omega) = 0$

Poincare lemma

M 是可縮的

$$(1) \int_I \omega \text{ 與路徑無關} \quad (2) d\omega = 0 \quad (3) \exists \varphi \text{ 使得 } \omega = d\varphi \quad \text{等價}$$

習作 誰怕向量微積分 p.138

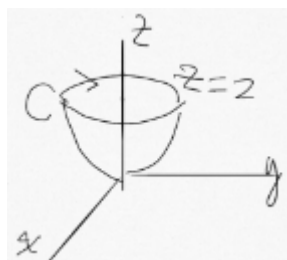
在電路 C 上的電動勢  $\varepsilon$  相當於電場 E 在此電路上的環積  $\varepsilon = \oint_C E \cdot tds$

法拉第發現在一穩定電路上  $\varepsilon = -\frac{1}{c} \frac{d\phi}{dt}$ ,  $\phi = \iint_S B \cdot \bar{n}dS$ , 試導出  $\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}$

$$\varepsilon = \oint_C E \cdot tds = \iint_S^{\text{Stokes}} (\nabla \times E) \cdot \bar{n}dS$$

$$\text{另一方面 } \varepsilon = -\frac{1}{c} \frac{d\phi}{dt} = -\frac{1}{c} \frac{d}{dt} \iint_S B \cdot \bar{n}dS = \iint_S \left( -\frac{1}{c} \frac{\partial B}{\partial t} \right) \cdot \bar{n}dS$$

$$\text{剝掉 } \iint \text{ 得 } \nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t}$$



$$A = [3y, -xz, yz^2]$$

$$S: z = x^2 + y^2$$

$$C: \begin{cases} x^2 + y^2 = 4 \\ z = 2 \end{cases}$$

Check Stokes 定理

$$\oint_C A \cdot tds = \iint_S (\nabla \times A) \cdot \bar{n} dS$$

左式是  $\int_{\partial D} w$ , 其中  $w = 3ydx - xzdy + yz^2dz$

$$\begin{cases} x = 2 \cos t \\ y = 2 \sin t, 0 \leq t \leq 2\pi, \text{ 則 } w = \dots = (-10 + 2 \cos 2t) dt \\ z = 2 \end{cases}$$

C 是順時鐘方向, 所以是  $\int_{2\pi}^0 (-10 + 2 \cos 2t) dt = 20\pi$

$$\text{右式 } \text{curl} A = \nabla \times A = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix} = [z^2 + x, 0, -z - 3]$$

$$z = f(x, y) = \frac{1}{2}(x^2 + y^2)$$

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx \wedge dy = \sqrt{1 + x^2 + y^2} dx \wedge dy$$

$$\bar{n} = \frac{\nabla(x^2 + y^2 - 2z)}{|\nabla(x^2 + y^2 - 2z)|} = \frac{1}{\sqrt{x^2 + y^2 + 1}} [x, y, -1]$$

所以  $\iint_S (\nabla \times A) \cdot \bar{n} dS = \iint_R (xz^2 + x^2 + z + 3) dx dy$ , let  $x = r \cos \theta, y = r \sin \theta$

$$= \int_0^{2\pi} \int_0^2 \left( \frac{1}{4} r^5 \cos \theta + r^2 \cos^2 \theta + \frac{r^2}{2} + 3 \right) r dr \wedge d\theta$$

$$= 20\pi$$

(注意  $dx \wedge dy = \frac{\partial(x, y)}{\partial(r, \theta)} dr \wedge d\theta = r dr \wedge d\theta$ , 其中  $\frac{\partial(x, y)}{\partial(r, \theta)} = \frac{\partial x}{\partial r} \frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial r} = r$ )

面積分 項 p. 367

$$D: \begin{cases} x = x(u, v) \\ y = y(u, v), \omega = P dx \wedge dy + Q dy \wedge dz + R dz \wedge dx \\ z = z(u, v) \end{cases}$$

$$\int_D \omega = \iint_D P dx \wedge dy + Q dy \wedge dz + R dz \wedge dx$$

$$= \iint_{D'} \left( P \frac{\partial(x,y)}{\partial(u,v)} + Q \frac{\partial(y,z)}{\partial(u,v)} + R \frac{\partial(z,x)}{\partial(u,v)} \right) du \wedge dv$$

其中  $\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$ , 以此類推

$$d\omega = \begin{vmatrix} dy \wedge dz & dz \wedge dx & dx \wedge dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

前頁就是在 check  $\int_{\partial D} \omega = \int_D d\omega$

靜電場

Handwritten mathematical derivation for electrostatics:

$$\vec{E}(r) = \frac{q}{r^2} \hat{r} \quad \vec{F} = q_0 \vec{E}$$

↓

$$\iint_S \vec{E} \cdot \vec{n} dS = 4\pi q \rightarrow \lim_{\Delta V \rightarrow 0} \frac{1}{\Delta V} \iint_S \vec{E} \cdot \vec{n} dS = 4\pi \rho$$

||  
dive

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 4\pi \rho$$

↓

$$\iint_S \vec{E} \cdot \vec{n} dS = \iiint_V \text{div } \vec{E} dV$$

↙

$$\oint_C \vec{F} \cdot \vec{t} ds = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS$$

↘

$$\oint_S \vec{E} \cdot \vec{t} dS = 0 \rightarrow \oint_S \vec{E} \cdot \vec{t} dS = 0$$

↓

$$\vec{E} = -\nabla \phi$$

↓

$$\nabla^2 \phi = -4\pi \rho$$

↙

$$\vec{n} \cdot \text{curl } \vec{E} = \lim_{\Delta S \rightarrow 0} \frac{1}{\Delta S} \oint_S \vec{E} \cdot \vec{t} ds = 0$$

↓

$$\nabla \times \vec{E} = 0$$

↘

$$\int_D d\omega = \int_{\partial D} \omega$$

Let  $x, y, z$  be the restrictions of the Cartesian coordinate functions in  $\mathbb{R}^3$  to  $S^2$ , oriented so that  $\{(1, 0, 0); (0, 1, 0)\}$  is a positively oriented basis of  $T_{(0,0,1)}S^2$ , and consider the 2-form

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2).$$

Compute the integral

$$\int_{S^2} \omega$$

using the parameterizations corresponding to

- (a) spherical coordinates;
- (b) stereographic projection.

$$\psi : [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \xrightarrow[\psi^*]{\psi} S^2$$

$$\psi(\theta, \varphi) = (\cos \theta \cos \varphi, \sin \theta \cos \varphi, \sin \varphi) \quad \text{Then}$$

$$\psi^* dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi = -\sin \theta \cos \varphi d\theta - \cos \theta \sin \varphi d\varphi$$

$$\psi^* dy = \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi = \cos \theta \cos \varphi d\theta - \sin \theta \sin \varphi d\varphi$$

$$\psi^* dz = \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi = \cos \varphi d\varphi$$

$$\begin{aligned} \psi^* \omega &= x(\psi^* dy \wedge \psi^* dz) + y(\psi^* dz \wedge \psi^* dx) + z(\psi^* dx \wedge \psi^* dy) \\ &= (\cos \theta \cos \varphi)(\cos \theta \cos^2 \varphi) d\theta \wedge d\varphi \\ &\quad + (\sin \theta \cos \varphi)(\sin \theta \cos^2 \varphi) d\theta \wedge d\varphi \\ &\quad + (\sin \varphi)[\sin^2 \theta \sin \varphi \cos \varphi + \cos^2 \theta \sin \varphi \cos \varphi] d\theta \wedge d\varphi \\ &= \cos \varphi d\theta \wedge d\varphi \end{aligned}$$

$$\int_{S^2} \omega = \int_U \psi^* \omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \cos \varphi d\theta d\varphi = 4\pi$$

$$d\omega = 3dx \wedge dy \wedge dz$$

$$\int_{S^2} \omega = \int_{\Omega} d\omega = 3 \int_{\Omega} dx \wedge dy \wedge dz = 4\pi \text{ 是單位球的 3 倍體積。}$$

這驗證了 Stokes 定理  $\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$  , 其中  $S^2 = \partial\Omega$  ,  $\Omega$  是實心球