§ Christoffel Symbols

設$[ij,k]=x_{ij}\cdot x_k$，則$[ij,k]=\frac{1}{2}\left\{\frac{\partial g_{jk}}{\partial u^i}+\frac{\partial g_{ik}}{\partial u^j}-\frac{\partial g_{ij}}{\partial u^k}\right\}$

引入g^{jk}，$g_{ij}\times g^{jk}=\delta_i^k$

高斯方程$x_{ij}=\Gamma_{ij}^k x_k+b_{ij}N$，兩邊對$x_l$作內積 則

$[ij,l]=\Gamma_{ij}^k g_{kl}$

，同乘g^{kl}得$\Gamma_{ij}^k=g^{kl}[ij,l]$

其中$A=\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix} E & F \\ F & G\end{pmatrix},A^{-1}=\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22}\end{pmatrix}$

曲面 z=f(x,y)的 Christoffel symbols

其中$p=\frac{\partial z}{\partial x},q=\frac{\partial z}{\partial y},r=\frac{\partial^2 z}{\partial x^2},s=\frac{\partial^2 z}{\partial x\partial y},t=\frac{\partial^2 z}{\partial y^2}$

$\Gamma_{11}^1=$ $\Gamma_{12}^1=$ $\Gamma_{22}^1=$

$\Gamma_{11}^2=$ $\Gamma_{12}^2=$ $\Gamma_{22}^2=$

$E=1+p^2,F=pq,G=1+q^2$，$EG-F^2=1+p^2+q^2,K=\frac{rt-s^2}{(1+p^2+q^2)^2}$

X=[x,y,z]

$X_1=[1,0,\frac{\partial z}{\partial x}]=[1,0,p]$，$X_2=[0,1,\frac{\partial z}{\partial y}]=[0,1,q]$

$X_{11}=[0,0,r],X_{12}=[0,0,s],X_{22}=[0,0,t]$

$[11,1]=x_{11}\cdot x_1=pr,[11,2]=x_{11}\cdot x_2=qr$

$\Gamma_{11}^1=g^{11}[11,1]+g^{12}[11,2]=\frac{1}{1+p^2+q^2}[(1+q^2)pr-pq(qr)]=\frac{pr}{1+p^2+q^2}$

以此類推

$\Gamma_{12}^1=\frac{ps}{1+p^2+q^2},\Gamma_{22}^1=\frac{pt}{1+p^2+q^2}$

$\Gamma_{11}^2=\frac{qr}{1+p^2+q^2},\Gamma_{12}^2=\frac{qs}{1+p^2+q^2},\Gamma_{22}^2=\frac{qt}{1+p^2+q^2}$