Lie derivative

[DG06]Ch3

Definition 3.1.1. A *local flow* is a smooth map $\Phi : \mathcal{N} \to M$, $(t, m) \mapsto \Phi^t(m)$, where $\mathcal{N}$ is a balanced neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that

(a) $\Phi^0(m) = m, \forall m \in M$.

(b) $\Phi^t(\Phi^s(m)) = \Phi^{t+s}(m)$ for all $s, t \in \mathbb{R}$, $m \in M$ such that

$$(s, m),\ (s + t, m),\ \ (t, \Phi^s(m)) \in \mathcal{N}.$$

When $\mathcal{N} = \mathbb{R} \times M$, Φ is called a *flow*. □

The conditions (a) and (b) above show that a flow is nothing but a left action of the additive (Lie) group $(\mathbb{R}, +)$ on M.

Example 3.1.2. Let A be a $n \times n$ real matrix. It generates a flow Φ^t_A on $\mathbb{R}^n$ by

$$\Phi^t_A x = e^{tA} x = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) x. \quad □$$

Definition 3.1.3. Let $\Phi : \mathcal{N} \to M$ be a local flow on M. The *infinitesimal generator* of Φ is the vector field X on M defined by

$$X(p) = X_\Phi(p) := \frac{d}{dt}|_{t=0}\ \Phi^t(p),\ \ \forall p \in M,$$

i.e., $X(p)$ is the tangent vector to the smooth path $t \mapsto \Phi^t(p)$ at $t = 0$. This path is called the *flow line* through p. □

Exercise 3.1.4. Show that X_Φ is a *smooth* vector field. □

Example 3.1.5. Consider the flow e^{tA} on $\mathbb{R}^n$ generated by a $n \times n$ matrix A. Its generator is the vector field X_A on $\mathbb{R}^n$ defined by

$$X_A(u) = \frac{d}{dt}|_{t=0}\ e^{tA} u = Au. \quad □$$

Proposition 3.1.6. *Let M be a smooth n-dimensional manifold. The map*

$$X : \{\text{Local flows on } M\} \to \text{Vect}\,(M),\quad \Phi \mapsto X_\Phi,$$

is a surjection. Moreover, if $\Phi_i : \mathcal{N}_i \to M$ (i=1,2) are two local flows such that $X_{\Phi_1} = X_{\Phi_2}$, then $\Phi_1 = \Phi_2$ on $\mathcal{N}_1 \cap \mathcal{N}_2$.

(書中有證明 略)

§ Lie derivative of a vector field

$$X=\sum_i X^i \frac{\partial}{\partial x^i}, Y=\sum_i Y^i \frac{\partial}{\partial x^i}$$

φ_t is the flow a vector field X，Y is a C^∞ vector field，then the Lie derivative of Y along X is $L_X Y=\lim_{t\to 0}\frac{(\varphi_{-t})_* Y - Y}{t}=\frac{d}{dt}((\varphi_{-t})Y)\big|_{t=0}$

(1) $L_X Y=[X,Y]=\sum_i (XY^i - YX^i)\frac{\partial}{\partial x^i}$

(2) $[L_X, L_Y]=L[X,Y]$

(3) $L_{aX+bY}=aL_X+bL_Y]$

(4) $L_X[Y,Z]=[L_XY,Z]+[Y,L_XZ]$

$$\nabla_X Y=\sum_i (XY^i+\sum_{j,k}\Gamma^i_{jk}X^jY^k)\frac{\partial}{\partial x^i}...\nabla_\mu V^\nu=\partial_\mu V^\nu+\Gamma^\nu_{\mu\rho}V^\rho$$

P1. ***R-bilineariiy.*** $\forall X,Y,Z\in \mathfrak{X}(M),\ \forall a,b\in R$

$$L_{X+Y}Z=L_XZ+L_YZ$$
$$L_X(Y+Z)=L_XY+L_XZ$$
$$L_{aX}bY=abL_XY.$$

P2. *Anticornmutative* **property.** $\forall X,Y\in\mathfrak{X}(M)$:

$$L_XY=-L_YX.$$

P3. *Jacobi identity.* $\forall X,Y,Z\in\mathfrak{X}(M)$:

$$[X,[Y,Z]]+[Y,[Z,X]]=-[Z,[X,Y]]$$
$$\Leftrightarrow\quad L_{[X,Y]}Z=L_XL_YZ-L_YL_XZ$$
$$\Leftrightarrow\quad L_X[Y,Z]=[L_XY,Z]+[Y,L_XZ].$$

So the properties P2 and P3 show the algebra is a Lie algebra. The last equality proves L_X is a Lie bracket derivation.

§ Lie derivative of a form ω

$X \in \chi(M)$, $L_X\omega := \lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega - \omega) = \frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0}$, Where φ_t is the local flow of X

The Lie derivative of forms can be expressed in terms of exterior derivative and interior product by Cartan's magic formulas：

For any $X \in \chi(M)$ and $\omega \in \Omega^p(M)$

$L_X\omega = \imath_X d\omega + d(\imath_X\omega)$

1. $L_X f = \langle df, X\rangle = Xf$, $f \in \Omega^0$
2. $L_X(df) = d(L_X f)$, $L_X d\omega = d(L_X\omega)$
3. $L_X(f\omega) = fL_X\omega + (L_X f)\omega$
4. $L_X(\omega\wedge\eta) = (L_X\omega)\wedge\eta + \omega\wedge(L_X\eta)$
5. $d(\omega+\eta) = d\omega + d\eta$
6. $L_X\varphi^*\omega = L_{\varphi^* X}\omega$
7. If ω is a k-form then $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^k\omega\wedge d\eta$
8. $d(d\omega) = 0$
9. $(L_X\omega)Y = L_X(\omega(Y)) - \omega([X,Y])$
10. $X, Y \in \chi(M)$ then $[L_X, L_Y] = L_{[X,Y]}$

例 $X = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y} + H\frac{\partial}{\partial z}$, volume form $dv = dx\wedge dy\wedge dz$

$L_X dx = d(L_X x) = d(Xx) = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$, Then

$L_X dv = L_X(dx\wedge dy\wedge dz) = (L_X dx)\wedge dy\wedge dz + dx\wedge(L_X dy)\wedge dz + dx\wedge dy\wedge(L_X dz)$

$= (\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z})dv = (divX)dv$

Prove of (4)

$L_X(\omega\wedge\eta) = \frac{d}{dt}\varphi_t^*(\omega\wedge\eta)\Big|_{t=0} = \frac{d}{dt}(\varphi_t^*\omega)\wedge(\varphi_t^*)\big|_{t=0}$

$= (\frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0})\wedge\eta + \omega\wedge(\frac{d}{t}(\varphi_t^*\eta)\big|_{t=0}) = (L_X\omega)\wedge\eta + \omega\wedge(L_X\eta)$

習作

1. 由(1)(2)(3)證明 $L_X(gdf)=(Xg)df+gd(Xf)$，where $f,g\in\Omega^0$

2. Let $\omega=\sum_i h_i dx^i, X=\sum_i \xi^i \frac{\partial}{\partial x^i}$，then $L_X\omega=\sum_j (Xh_j)dx^j+\sum_k h_k d\xi^k$

   (Let $\omega=\omega_i d\omega^i, X=X^j\frac{\partial}{\partial x^j}$，then $L_X\omega=(X^j\frac{\partial \omega_k}{\partial x^j}+\omega_j\frac{\partial x^j}{\partial x^k})dx^k$)

3. P.77 lemma 證明 $L_XY=[X,Y]$

4. P.80 Corollary 證明 $[L_X,L_Y]=L_{[X,Y]}$

5. P.80 證明 $L_X(\omega\wedge\eta)=(L_X\omega)\wedge\eta+\omega\wedge(L_X\eta)$

Lemma

For any differential form $\omega\in\Omega^1(M)$，and any vector fields X ,Y，we have $(L_X\omega)Y=L_X(\omega(Y))-\omega([X,Y])$

Proof. Denote by Φ^t the local flow generated by X. We have $\Phi_*^t\omega=(\Phi^{-t})^*\omega$, i.e., for any $p\in M$, and any $Y\in \mathrm{Vect}\,(M)$, we have

$$(\Phi_*^t\omega)_p(Y_p)=\omega_{\Phi^{-t}p}\big(\Phi_*^{-t}Y_p\big).$$

Fix a point $p\in M$, and choose local coordinates (x^i) near p. Then

$$\omega=\sum_i \omega^i dx^i,\quad X=\sum_i X^i\frac{\partial}{\partial x^i},\quad Y=\sum_i Y^i\frac{\partial}{\partial x^i}.$$

Denote by $\gamma(t)$ the path $t\mapsto\Phi^t(p)$. We set $\omega_i(t)=\omega_i(\gamma(t))$, $X_0^i=X^i(p)$, and $Y_0^i=Y^i(p)$. Using (3.1.6) we deduce

$$(\Phi_*^t\omega)_p(Y_p)=\sum_i\omega_i(-t)\cdot\Big(Y_0^i-t\sum_j\frac{\partial X^i}{\partial x^j}Y_0^j+O(t^2)\Big).$$

Hence

$$-(L_X\omega)Y=\frac{d}{dt}|_{t=0}(\Phi_*^t\omega)_p(Y_p)=-\sum_i\dot\omega_i(0)Y_0^i-\sum_{i,j}\omega_i(0)\frac{\partial X^i}{\partial x^j}Y_0^j.$$

One the other hand, we have

$$X\cdot\omega(Y)=\frac{d}{dt}|_{t=0}\sum_i\omega_i(t)Y^i(t)=\sum_i\dot\omega_i(0)Y_0^i+\sum_{i,j}\omega_i(0)X_0^j\frac{\partial Y^i}{\partial x^j}.$$

We deduce that

$$X\cdot\omega(Y)-(L_X\omega)Y=\sum_{i,j}\omega_i(0)\Big(X_0^j\frac{\partial Y^i}{\partial x^j}-\frac{\partial X^i}{\partial x^j}Y_0^j\Big)=\omega_p([X,Y]_p).\qquad\square$$

Example 3.1.17. Let $\omega = \omega_i dx^i$ be a 1-form on $\mathbb{R}^n$. If $X = X^j \frac{\partial}{\partial x^j}$ is a vector field on $\mathbb{R}^n$ then $L_X\omega = (L_X\omega)_k dx^k$ is defined by

$$(L_X\omega)_k = (L_X\omega)(\frac{\partial}{\partial x_k}) = X\omega(\frac{\partial}{\partial x_k}) - \omega(L_X\frac{\partial}{\partial x_k}) = X\cdot\omega_k + \omega\left(\frac{\partial X^i}{\partial x^k}\frac{\partial}{\partial x_i}\right).$$

Hence

$$L_X\omega = \left(X^j\frac{\partial \omega_k}{\partial x^j} + \omega_j\frac{\partial X^j}{\partial x^k}\right)dx^k.$$

In particular, if $X = \partial_{x^i} = \sum_j \delta^{ij}\partial_{x^j}$, then

$$L_X\omega = L_{\partial_{x^i}}\omega = \sum_{k=1}^{n}\frac{\partial\omega_k}{\partial x^i}dx^k.$$

If X is the radial vector field $X = \sum_i x^i\partial_{x^i}$, then

$$L_X\omega = \sum_k (X\cdot\omega_k + \omega_k)dx^k. \qquad \square$$

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1.12. **Interior product and Cartan's formula.** If X is a vector field we define the *interior product*

$$\iota_X : \Omega^*(M) \to \Omega^{*-1}(M)$$

for each p by contaction of tensors. In other words, if $\omega \in \Omega^p$ then

$$(\iota_X\omega)(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1})$$

for any $X_1, \cdots, X_{p-1} \in \mathfrak{X}(M)$. On 1-forms this reduces to $\iota_X(\alpha) = \alpha(X)$.

Proposition 1.29 (Properties of ι). *Interior product satisfies $\iota_X\iota_Y\omega = -\iota_Y\iota_X\omega$ and the Leibniz rule*

$$\iota_X(\alpha\wedge\beta) = (\iota_X\alpha)\wedge\beta + (-1)^p\alpha\wedge(\iota_X\beta)$$

whenever $\alpha \in \Omega^p$.

習作

1. 證明 $L_X Y = [X, Y]$
2. Prove that $[L_X, L_Y]$ is a derivative on the algebra $C^\infty(M)$ and

   $L_{[X,Y]} = [L_X, L_Y]$
3. $L_X Y = -L_Y X$