[DG06]Ch3

Definition 3.1.1. A local flow is a smooth map $\Phi : N \to M$, $(t, m) \mapsto \Phi^t(m)$, where $N$ is a balanced neighborhood of $\{0\} \times M$ in $\mathbb{R} \times M$, such that

(a) $\Phi^0(m) = m$, $\forall m \in M$.

(b) $\Phi^t(\Phi^s(m)) = \Phi^{t+s}(m)$ for all $s, t \in \mathbb{R}$, $m \in M$ such that $(s, m), (s + t, m), (t, \Phi^s(m)) \in N$.

When $N = \mathbb{R} \times M$, $\Phi$ is called a flow.

The conditions (a) and (b) above show that a flow is nothing but a left action of the additive (Lie) group $(\mathbb{R}, +)$ on $M$.

Example 3.1.2. Let $A$ be a $n \times n$ real matrix. It generates a flow $\Phi^t_A$ on $\mathbb{R}^n$ by

$$\Phi^t_A x = e^{tA} x = \left( \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \right) x.$$

Definition 3.1.3. Let $\Phi : N \to M$ be a local flow on $M$. The infinitesimal generator of $\Phi$ is the vector field $X$ on $M$ defined by

$$X(p) = X_\Phi(p) := \frac{d}{dt} \bigg|_{t=0} \Phi^t(p), \quad \forall p \in M,$$

i.e., $X(p)$ is the tangent vector to the smooth path $t \mapsto \Phi^t(p)$ at $t = 0$. This path is called the flow line through $p$.

Exercise 3.1.4. Show that $X_\Phi$ is a smooth vector field.

Example 3.1.5. Consider the flow $e^{tA}$ on $\mathbb{R}^n$ generated by a $n \times n$ matrix $A$. Its generator is the vector field $X_A$ on $\mathbb{R}^n$ defined by

$$X_A(u) = \frac{d}{dt} \bigg|_{t=0} e^{tA} u = A u.$$

Proposition 3.1.6. Let $M$ be a smooth $n$-dimensional manifold. The map

$$X : \{\text{Local flows on } M\} \to \text{Vect}(M), \quad \Phi \mapsto X_\Phi,$$

is a surjection. Moreover, if $\Phi_i : N_i \to M$ ($i=1,2$) are two local flows such that $X_{\Phi_1} = X_{\Phi_2}$, then $\Phi_1 = \Phi_2$ on $N_1 \cap N_2$. 

(書中有證明 略)
\[ X = \sum_i X^i \frac{\partial}{\partial x^i}, Y = \sum_i Y^i \frac{\partial}{\partial x^i} \]

(1) \[ L_X Y = [X, Y] = \sum_i (XY^i - YX^i) \frac{\partial}{\partial x^i} \]

(2) \[ \nabla_X Y = \sum_i (XY^i + \sum_{j,k} \Gamma^i_{jk} X^j Y^k) \frac{\partial}{\partial x^i}, \ldots \]

\[ \nabla^\mu V^\nu = \partial^\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho \]

\[ X \in \chi(M), \quad L_X \omega = \lim_{t \to 0} \frac{1}{t} (\varphi_t^* \omega - \omega) = \left. \frac{d}{dt} (\varphi_t^* \omega) \right|_{t=0}, \quad \text{Where} \quad \varphi_t \quad \text{is the local flow of} \quad X \]

The Lie derivative of forms can be expressed in terms of exterior derivative and interior product by Cartan’s “magic formula”:

**Proposition 1.31 (Cartan’s Magic Formula).** For any \( X \in \chi(M) \) and \( \alpha \in \Omega^p(M) \) there is an identity \( \mathcal{L}_X (\alpha) = \iota_X (d\alpha) + d(\iota_X (\alpha)) \). In other words, as operators on forms, \( \mathcal{L}_X = \iota_X d + d\iota_X \)

1. \( L_X f = < df, X > \cdot Xf, \quad f \in \Omega^0 \)
2. \( L_X (df) = d(L_X f), \quad L_X d\omega = d(L_X \omega) \)
3. \( L_X (\omega \wedge \eta) = (L_X \omega) \wedge \eta + \omega \wedge (L_X \eta) \)
4. \( d(\omega + \eta) = d\omega + d\eta \)
5. If \( \omega \) is a k-form then \( d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \)
6. \( d(d\omega) = 0 \)
7. \( (L_X \omega)Y = L_X (\omega(Y)) - \omega([X, Y]) \)
8. \( X, Y \in \chi(M) \) then \( [L_X, L_Y] = L_{[X,Y]} \)

例

\[ X = F \frac{\partial}{\partial x} + G \frac{\partial}{\partial y} + H \frac{\partial}{\partial z}, \quad \text{volume form} \quad dv = dx \wedge dy \wedge dz \]

\[ L_X dx = d(L_X x) = d(Xx) = dF = \frac{\partial F}{\partial x} \; dx + \frac{\partial F}{\partial y} \; dy + \frac{\partial F}{\partial z} \; dz, \quad \text{Then} \]

\[ L_X dv = L_X (dx \wedge dy \wedge dz) = (L_X dx) \wedge dy \wedge dz + dx \wedge (L_X dy) \wedge dz + dx \wedge dy \wedge (L_X dz) \]

\[ = (\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}) dv = (\text{div} X) dv \]
Prove of (3)

\[ L_x (\omega \wedge \eta) = \frac{d}{dt} (\varphi_t^*(\omega \wedge \eta)) \bigg|_{t=0} = \frac{d}{dt} (\varphi_t^* \omega \wedge (\varphi_t^* \eta)) \bigg|_{t=0} \]

\[ = (\frac{d}{dt} (\varphi_t^* \omega)) \bigg|_{t=0} \wedge \eta + \omega \wedge (\frac{d}{dt} (\varphi_t^* \eta)) \bigg|_{t=0} = (L_x \omega) \wedge \eta + \omega \wedge (L_x \eta) \]

習作

1. 由(1)(2)(3)證明 \( L_x (gdf) = (Xg)df + gd(\omega f) \) ，其中 \( f, g \in \Omega^0 \)

2. Let \( \omega = \sum h_i dx^i, X = \sum \xi^i \frac{\partial}{\partial x^i} \)，則 \( L_x \omega = \sum (Xh_i) dx^i + \sum h_i d\xi^k \)

(Let \( \omega = \omega_i dx^i, X = X^i \frac{\partial}{\partial x^j} \)，則 \( L_x \omega = (X^i \frac{\partial \omega_j}{\partial x^i} + \omega_j \frac{\partial x_i}{\partial x^j}) dx^i \) )

3. P.77 lemma 證明 \( L_X Y = [X, Y] \)

4. P.80 Corollary 證明 \([L_X, L_Y] = L_{[X, Y]} \)

5. P.80 證明 \( L_x (\omega \wedge \eta) = (L_x \omega) \wedge \eta + \omega \wedge (L_x \eta) \)
Lemma 3.1.11. For any differential form \( \omega \in \Omega^1(M) \), and any vector fields \( X, Y \in Vect(M) \)
we have
\[
(L_X \omega)(Y) = L_X (\omega(Y)) - \omega([X, Y]),
\]  
(3.1.8)
where \( X \cdot \omega(Y) \) denotes the (Lie) derivative of the function \( \omega(Y) \) along the vector field \( X \).

Proof. Denote by \( \Phi^t \) the local flow generated by \( X \). We have \( \Phi^t_\ast \omega = (\Phi^{-t})^\ast \omega \), i.e., for any \( p \in M \), and any \( Y \in Vect(M) \), we have
\[
(\Phi^t_\ast \omega)_p(Y_p) = \omega_{\Phi^t_p} - t \omega_{\Phi^t_p}(\Phi^{-t}_p Y_p).
\]
Fix a point \( p \in M \), and choose local coordinates \( (x^i) \) near \( p \). Then
\[
\omega = \sum_i \omega^i dx^i, \quad X = \sum_i X^i \frac{\partial}{\partial x^i}, \quad Y = \sum_i Y^i \frac{\partial}{\partial x^i}.
\]
Denote by \( \gamma(t) \) the path \( t \mapsto \Phi^t(p) \). We set \( \omega_i(t) = \omega_i(\gamma(t)) \), \( X^i_0 = X^i(p) \), and \( Y^i_0 = Y^i(p) \).

Using (3.1.6) we deduce
\[
(\Phi^t_\ast \omega)_p(Y_p) = \sum_i \omega_i(t) \left( Y^i_0 - t \sum_j \frac{\partial X^j}{\partial x^i} Y^j_0 + O(t^2) \right) - t \sum_i \omega_i(t) \frac{\partial X^i}{\partial x^j} Y^j_0.
\]
Hence
\[
-(L_X \omega)Y = \lim_{t \to 0} \frac{d}{dt}(\Phi^t_\ast \omega)_p(Y_p) = \sum_i \omega_i(0) Y^i_0 - \sum_{i,j} \omega_i(0) \frac{\partial X^i}{\partial x^j} Y^j_0.
\]
One the other hand, we have
\[
X \cdot \omega(Y) = \lim_{t \to 0} \frac{d}{dt} \sum_i \omega_i(t) Y^i(t) = \sum_i \omega_i(0) Y^i_0 + \sum_{i,j} \omega_i(0) X^i_0 \frac{\partial Y^j}{\partial x^i}.
\]
We deduce that
\[
X \cdot \omega(Y) - (L_X \omega)Y = \sum_{i,j} \omega_i(0) \left( X^i_0 \frac{\partial Y^j}{\partial x^i} - \frac{\partial X^i}{\partial x^j} Y^j_0 \right) = \omega_p([X, Y]_p). \quad \square
\]

Example 3.1.17. Let \( \omega = \omega_i dx^i \) be a 1-form on \( \mathbb{R}^n \). If \( X = X^j \frac{\partial}{\partial x^j} \) is a vector field on \( \mathbb{R}^n \) then
\[
L_X \omega = (L_X \omega)_k dx^k \text{ is defined by}
\]
\[
(L_X \omega)_k = (L_X \omega) \left( \frac{\partial}{\partial x_k} \right) = X^j \frac{\partial \omega_k}{\partial x_j} - \omega_j \frac{\partial X^j}{\partial x_k} = X \cdot \omega_k + \omega \left( \frac{\partial X^j}{\partial x_k} \frac{\partial}{\partial x_j} \right).
\]
Hence
\[
L_X \omega = \left( X^j \frac{\partial \omega_k}{\partial x_j} + \omega_j \frac{\partial X^j}{\partial x_k} \right) dx^k.
\]
In particular, if \( X = \partial_x^i = \sum_j \delta^i_j \partial_x^j \), then
\[
L_X \omega = L_{\partial_x^i} \omega = \sum_{k=1}^n \frac{\partial \omega_k}{\partial x^i} dx^k.
\]
If \( X \) is the radial vector field \( X = \sum_i x^i \partial_x^i \), then
\[
L_X \omega = \sum_k \left( X \cdot \omega_k + \omega_k \right) dx^k. \quad \square
\]
1.12. **Interior product and Cartan’s formula.** If $X$ is a vector field we define the \textit{interior product} 

$$\iota_X : \Omega^p(M) \to \Omega^{n-1}(M)$$

for each $p$ by contaction of tensors. In other words, if $\omega \in \Omega^p$ then 

$$(\iota_X \omega)(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1})$$

for any $X_1, \cdots, X_{p-1} \in \mathfrak{X}(M)$. On 1-forms this reduces to $\iota_X(\alpha) = \alpha(X)$.

**Proposition 1.29** (Properties of $i$). \textit{Interior product satisfies} $\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$ \textit{and the Leibnitz rule} 

$$\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_X \beta)$$

\textit{whenever} $\alpha \in \Omega^p$. 

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The exponential map of a Lie group & exterior derivative 暫略