可微分流形的背景是廣義相對論，時空是一個四維黎曼流形(M ,g)。
重力是時空的曲率形成的，而曲率來自物質。
物理性質與座標無關(座標變換下的不變量 張量)，我們只能定義局部座標，所以流形是局部歐氏空間。

§ Topological Manifold

(i) M is **Hausdorff**, that is, for each pair p_1, p_2 of distinct points of M there exist neighborhoods V_1, V_2 of p_1 and p_2 such that $V_1 \cap V_2 = \varnothing$.
(ii) Each point $p \in M$ possesses a neighborhood V homeomorphic to an open subset U of $\mathbb{R}^n$.
(iii) M satisfies the **second countability axiom**, that is, M has a countable basis for its topology.

Example 1.3

(1) Every open subset M of $\mathbb{R}^n$ with the subspace topology (that is, $U \subset M$ is an open set if and only if $U = M \cap V$ with V an open set of $\mathbb{R}^n$) is a topological manifold.
(2) (*Circle*) The **circle**

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\right\}$$

with the subspace topology is a topological manifold of dimension 1. Conditions (i) and (iii) are inherited from the ambient space. Moreover, for each point $p \in S^1$ there is at least one coordinate axis which is not parallel to the vector n_p normal to S^1 at p. The projection on this axis is then a homeomorphism between a (sufficiently small) neighborhood V of p and an interval in $\mathbb{R}$.

(3) (2-*sphere*) The previous example can be easily generalized to show that the 2-**sphere**

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$$

with the subspace topology is a topological manifold of dimension 2.
(4) (*Torus of revolution*) Again, as in the previous examples, we can show that the surface of revolution obtained by revolving a circle around an axis that does not intersect it is a topological manifold of dimension 2 (Fig. 1.1).
(5) The surface of a cube is a topological manifold (homeomorphic to S^2).

(1) (*Torus*) The **torus** T^2 is the quotient of the unit square $Q = [0, 1]^2 \subset \mathbb{R}^2$ by the equivalence relation

$$(x, y) \sim (x + 1, y) \sim (x, y + 1),$$

equipped with the quotient topology (cf. Sect. 1.10.1).

(2) (*Klein bottle*) The **Klein bottle** K^2 is the quotient of Q by the equivalence relation

$$(x, y) \sim (x + 1, y) \sim (1 - x, y + 1).$$

(3) (*Projective plane*) The **projective plane** $\mathbb{R}P^2$ is the quotient of Q by the equivalence relation

$$(x, y) \sim (x + 1, 1 - y) \sim (1 - x, y + 1).$$

Exercise 1.8

(1) Which of the following sets (with the subspace topology) are topological manifolds?

(a) $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$;
(b) $S^2 \setminus \{p\}$ $(p \in S^2)$;

(c) $S^2 \setminus \{p, q\}$ $(p, q \in S^2, p \neq q)$;
(d) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$;
(e) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$;

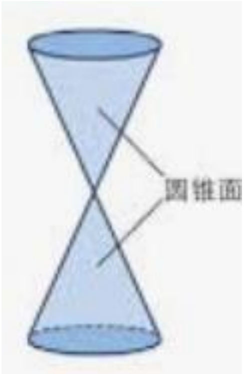
(2) Which of the manifolds above are homeomorphic?
(3) Show that the Klein bottle K^2 can be obtained by gluing two Möbius bands together through a homeomorphism of the boundary.
(4) Show that:

(a) $M\#S^2 = M$ for any 2-dimensional topological manifold M;
(b) $\mathbb{R}P^2\#\mathbb{R}P^2 = K^2$;
(c) $\mathbb{R}P^2\#T^2 = \mathbb{R}P^2\#K^2$.

(a) adding a vertex to a triangulation does not change $\chi(M)$;
(b) $\chi\left(S^2\right) = 2$;
(c) $\chi\left(T^2\right) = 0$;
(d) $\chi\left(K^2\right) = 0$;
(e) $\chi\left(\mathbb{R}P^2\right) = 1$;
(f) $\chi(M\#N) = \chi(M) + \chi(N) - 2$.

註 Ex1-8 的解答在 p.330

$S = \{(x, y, z) \in R^3 \,|\, x^2 + y^2 = z^2\}$ 不是拓樸流形



若 S 是一 topological manifold

$W \subset S$ 是一包含原點的 connected open set $\cong U \subset R^2$

則 $W-\{o\} \cong U-\{q\}$，其中 q 是 preimage of O

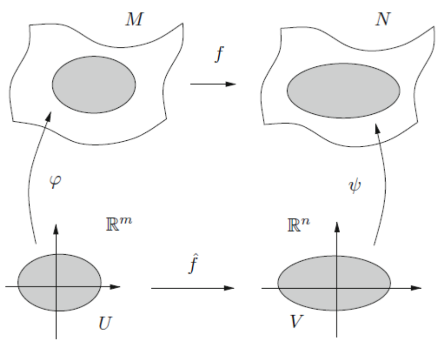
但是 W-{O} 不連通，U-{q}連通

矛盾。

說明 $M = S^2 - \{p\}(p \in S^2)$ 是一個 2-dim topological manifold

$\varphi : R^2 \cong C \to M$

$\varphi(z) = (\frac{2x}{1+|z|^2}, \frac{2y}{1+|z|^2}, \frac{|z|-1}{1+|z|^2})$，其中 z=x +i y



f is a differentiable map

$\hat{f} := \psi^{-1} \circ f \circ \varphi : U \subset R^m \to R^n$ is smooth

F is called diffeomorphism(微分同胚)

若

(1) f is bijective(對射)

(2) f^{-1} also differentiable

習作

1. Consider the orthogonal group： $O(n) = \{g \in GL(n;R) : g^{-1} = g^T\}$

   Show that O(n) is a dfferentiable manifold，and determine its dimension。

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$\dim(O(n)) = \frac{n(n-1)}{2}$

2. Let $X = \{(x, y, z) \in R^3 \,|\, x^3 + xyz + y^2 = 1\}$

   (a) Show that X is a 2-dim manifold

   (b) Consider the map $\pi : X \to R^2$ taking (x,y,z) to (x,y)

   Find all points of X at which π fails to be a local diffeomorphism

3.