第三章 Curvature

(1) Overview

黎曼流形(M,g) 先給定度量，演算五個基本的式子：

(1) Christoffel symbol $\Gamma^i_{jk} = \frac{1}{2} g^{il} (\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l})$

(2) Covariant derivative ∇

(3) Equation of a geodesic

(4) Riemannian curvature $R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x^i} - \frac{\partial \Gamma^l_{ik}}{\partial x^j} + \sum_m \Gamma^m_{jk} \Gamma^l_{im} - \sum_m \Gamma^m_{ik} \Gamma^l_{jm}$

(5) Einstein equation $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

(2) Covariant derivatives ∇

(M,g) is a Riemann manifold，then there is a unique symmetric connection ∇ on TM compatiable with the metric。

The covariant derivative operator ∇ perform the partial derivative，but in a way independent of coordinates。

1. Linearity ∇(T+S)=∇T+∇S
2. Leibniz rule $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$
   $\nabla_X (fY) = (X \cdot f)Y + f \nabla_X Y$
3. Commutes with contractions
4. $\nabla_\mu \phi = \partial_\mu \phi$ for a scalar ϕ

Torsion-free $\nabla_X Y - \nabla_Y X = [X,Y]$; $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$

Metric compatibility $X \cdot \langle Y,Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$; $\nabla_\rho g_{\mu\nu} = 0$

When ∇ compatible with metric and torsion free then ∇ is called Levi-Civita connection(Riemannian connection)。

$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$ the covariant derivative of a vector。

$\nabla_{\mu}\omega_{\nu}=\partial_{\mu}\omega_{\nu}-\Gamma^{\lambda}_{\mu\nu}\omega_{\lambda}$ the covariant derivative of a one-form。

Covariant derivative of Y along X

$X=\sum_{i}X^{i}\frac{\partial}{\partial x^{i}}$ 、 $Y=\sum_{i}Y^{i}\frac{\partial}{\partial x^{i}}$ then $\nabla_{X}Y=\sum_{i}(XY^{i}+\sum_{j,k}\Gamma^{i}_{jk}X^{j}Y^{k})\frac{\partial}{\partial x^{i}}$

Covariant derivative of ω along X

$\nabla_{X}\omega=\sum_{i}(X\omega^{i}-\sum_{jk}\Gamma^{k}_{ji}X^{j}\omega_{k})dx^{i}$

The covariant divergence of V^{μ} is given by $\nabla_{\mu}V^{\mu}=\partial_{\mu}V^{\mu}+\Gamma^{\mu}_{\mu\lambda}V^{\lambda}$

Show that $\Gamma^{\mu}_{\mu\lambda}=\frac{1}{\sqrt{|g|}}\partial_{\mu}(\sqrt{|g|}V^{\mu})$ ，and Stokes theorem in curved-space：

$$\int_{\Sigma}\nabla_{\mu}V^{\mu}\sqrt{|g|}d^{n}x=\int_{\partial\Sigma}n_{\mu}V^{\mu}\sqrt{|\gamma|}dx^{n-1}x$$

Where n_{μ} is normal to $\partial\Sigma$ ，and γ_{ij} is the induced metric on $\partial\Sigma$ 。

(3) Parallel tranport and Geodesics

The covariant derivative quantifies the instantaneous rate of change of a tensor field in comparison to what the tensor would be if it were "parallel transported" 。

Moving a vector along a path，keeping constant all the while，is known as parallel transport。

平行移動與共變微分相互決定。

$\frac{d}{d\lambda}V^{\mu}+\Gamma^{\mu}_{\sigma\rho}\frac{dx^{\sigma}}{d\lambda}V^{\rho}=0$ The equation of parallel transport for a vector。

The concept of moving a vector along a path，keeping constant all the while，is known as parallel transport。

The notion of parallel transport is obviously dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, the metric is always parallel transported with respect to it:

$$\frac{D}{d\lambda}g_{\mu\nu} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma g_{\mu\nu} = 0. \tag{3.41}$$

It follows that the inner product of two parallel-transported vectors is preserved. That is, if V^μ and W^ν are parallel-transported along a curve $x^\sigma(\lambda)$, we have

$$\begin{aligned}\frac{D}{d\lambda}(g_{\mu\nu}V^\mu W^\nu) &= \left(\frac{D}{d\lambda}g_{\mu\nu}\right)V^\mu W^\nu + g_{\mu\nu}\left(\frac{D}{d\lambda}V^\mu\right)W^\nu + g_{\mu\nu}V^\mu\left(\frac{D}{d\lambda}W^\nu\right)\\ &= 0.\end{aligned} \tag{3.42}$$

This means that parallel transport with respect to a metric-compatible connection preserves the norm of vectors, the sense of orthogonality, and so on.

§ geodesics

在廣義相對論中，一個自由粒子走的路徑(世界線)是測地線，此時黎曼流形的度量決定了空間的曲率。

等價(效)原理(equivalence principle)指出重力場決定伽利略時空的曲線，這些曲線是對稱聯絡(Cartan connection)的測地線。

A geodesic is a curve along which the tangent vector is parallel-transported。

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$$ or alternatively

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0$$

測地線的求法：

(1) 古典微分幾何
(2) 能量的 Euler-Lagrange 方程
(3) 從平行移動 協變微分
(4) 古典力學的最小耦合原理(minimal-coupling principle)
(5) 從熱力學 heat flow 方程看測地線
(6) 變分法

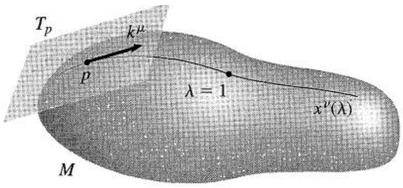
(4) Properties of Geodesics

Proper time：

An important property of geodesics in spacetime with Lorentzian metric is that the character (timelike/null/spacelike) of the geodesic，relative to a metric-compatible connection，never changes。

This is simply because parallel transport preserves inner products，and the character is determined by the inner product of the tangent vector with itself。

Defining an exponential map：(by means of the geodesic)



$$\exp_p : T_p \to M$$

The exponential map takes a vector in T_p to a point in M that lies at unit affine parameter along the geodesic to which the vector is tangent。

The singularity theorem of Hawking and Penrose：

For certain matter content，spacetime in general relativity are almost guaranteed to be geodesically incomplete。

Riemann normal coordinates at p：

$$\partial_\sigma g_{\mu\nu} = 0$$

Provide a realization of locally inertial coordinates。

(5) The Expanding Universe Revivisted

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

後面有用變分法求 geodesic，而後求 Christoffel symbols 的方法。

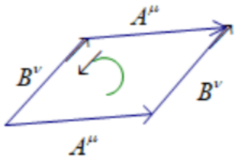
The cosmological redshift：

The wavelength of the photon is inversely proportional to the frequency，and in an expanding universe the wavelength therefore grows with time。

The universe is expanding，and that expansion stretches light traveling through space in a phenomenon known as cosmological redshift。

The greater the redshift，the greater the distance the light has traveled。

(6) The Riemann Curvature Tensor



向量V^{μ}沿$A^{\mu} \to B^{\nu} \to A^{\mu} \to B^{\nu}$(無窮小的 loop)的方向平行移動 回原點的改變量

$$\delta V^{\rho} = R^{\rho}_{\sigma\mu\nu} V^{\sigma} A^{\mu} B^{\nu}$$

Riemann curvature tensor， Ricci curvature tensor

定義 $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$

$$R^{l}_{ijk} = \frac{\partial \Gamma^{l}_{jk}}{\partial x^{i}} - \frac{\partial \Gamma^{l}_{ik}}{\partial x^{j}} + \sum_m \Gamma^{m}_{jk} \Gamma^{l}_{im} - \sum_m \Gamma^{m}_{ik} \Gamma^{l}_{jm}$$ Riemann tensor

$$R_{ijkl} = \sum_m R^{m}_{ijk} g_{ml}$$ curvature tensor

$$R_{ij} = \sum_k R^{k}_{ikj}$$ Ricci curvature tensor

$$R = g^{\mu\nu} R_{\mu\nu}$$ scalar curvature

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$ Eintein tensor

(7) Properties of the Riemann Tensor

The curvature R of a Riemannian manifold has the following properties：

1. $R(fX_1+gX_2,Y_1)=fR(X_1,Y_1)+gR(X_2,Y_1)$
   $R(X_1,fY_1+gY_2)=fR(X_1,Y_1)+gR(X_1,Y_2)$
2. $R(X,Y)(Z+W)=R(X,Y)Z+R(X,Y)W$
   $R(X,Y)fZ=fR(X,Y)Z$
3. R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=0 Bianchi identity $R_{ijkl}+R_{iklj}+R_{iljk}=0$
4. $R_{ijkl}=-R_{ijlk}=-R_{jikl}=R_{klij}$

(8) Symmetries and Killing Vectors

Noether 定理 1915 年：守恆律來自系統對稱。

Every differentiable symmetry of the action of a physical system with conservative forces has a corresponding conservative law。

A manifold M posseses a symmetry if the geometry is invariant under a certain trnsfomation that maps M to itself。

Symmetries of the metric are called isometries。

對稱性自發破缺：對稱性的破壞是事物不斷發展進化，世界變得豐富多彩的原因。

§ Killing vector field

Let $x:W\to R^n$ be local coordinates on open set $W\subset M$ 。

We take $\omega=\sum_i \omega_i dx^i$ ，then $\nabla_X\omega=\sum_i (X\cdot\omega_i-\sum_{j,k}\Gamma^k_{ji}X^j\omega_k)dx^i$

Let (M g) be a Riemannian manifold with Levi-Civita connection ∇ ，and let

$\psi_t:M\to M$ be a 1-parameter group of isometries。

The vector field $X \in \chi(M)$ defined by $X_p := \frac{d}{dt}_{t=0} \psi_t(p)$ is called Killing vector field associated to ψ_t。

或者直接以 $L_X g = 0$ (Killing equation)為定義。

Killing vector fields are used to discuss the isometries in GR。

A vector fields on (M,g) that preserve the metric。

The flows generated by Killing fields are continuous isometries on the manifold。

Example

In R^3 with metric $ds^2 = dx^2 + dy^2 + dz^2$，independent of the metric components w.r.t. x，y，and z immediately yields three Killing vectors：

$X^\mu = (1,0,0), Y^\mu = (0,1,0), Z^\mu = (0,0,1)$

These clearly represent the three translations。

For polar coordinates，$x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$

$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\phi^2$

Now the metric (the *same* metric, just in a different coordinate system) is manifestly independent of ϕ. We therefore know that $R = \partial_\phi$ is a Killing vector. Transforming back to Cartesian coordinates, this becomes

$$R = -y\partial_x + x\partial_y. \qquad (3.185)$$

The Cartesian components R^μ are therefore $(-y, x, 0)$. Since this represents a rotation about the z-axis, it is straightforward to guess the components of all three rotational Killing vectors:

$R^\mu = (-y, x, 0), S^\mu = (z, 0, -x), T^\mu = (0, -z, y)$ representing rotations about the z，y，and x-axes，respectively。

The Killing vectors for the two-sphere S^2 with metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$

Since this sphere can be thought of as the locus of points at unit distance from the origin in $\mathbf{R}^3$, and the rotational Killing vectors all rotate such a sphere into itself, they also represent symmetries of S^2. To get explicit coordinate-basis representations for these vectors, we first transform the three-dimensional vectors (3.186) to polar coordinates $x^{\mu'} = (r, \theta, \phi)$. A straightforward calculation reveals

$$R=\partial_{\phi}, S=\cos\phi\partial_{\theta}-\cot\theta\sin\phi\partial_{\phi}, T=-\sin\phi\partial_{\theta}-\cot\theta\cos\phi\partial_{\phi}$$

The three rotational Killing vectors in R^3 are exactly the same as those of S^2 in spherical polar coordinates。

(9) Maximally Symmetric Spaces

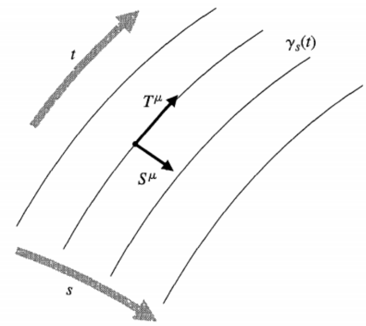
Maximally symmetric Euclidean spaces of negative curvature are hyperboloids H^n 。

Poincare half plane：

Gauss-Bonnet theorem $\chi(M)=\frac{1}{4\pi}\int_M R\sqrt{|g|}d^n x$

(10) Geodesic Deviation

We consider a one-parameter family of geodesics $\gamma_s(t)$ 。



A set of geodesics $\gamma_s(t)$，with tangent vector T^μ 。

The vector field S^μ measures the deviation between nearby geodesics。

$$T^\mu=\frac{\partial x^\mu}{\partial t} \text{，} S^\mu=\frac{\partial x^\mu}{\partial s}$$

We define the "relative velosity of geodesics" ：$V^\mu=(\nabla_T S)^\mu=T^\rho\nabla_\rho S^\mu$ and

"Relative acceleraion of geodesics" ：$A^\mu=(\nabla_T V)^\mu=T^\rho\nabla_\rho V^\mu$\

$T^{\alpha}\nabla_{\alpha}=\frac{D}{d\tau}$ ，and A^{μ}=taking the directional covariant derivative of X along T twice $=T^{\alpha}\nabla_{\alpha}(T^{\beta}\nabla_{\beta}X^{\mu})$

Since S and T are basis vectors adapted to a coordinates system，then [S,T]=0

(torsion free， $\nabla_X Y-\nabla_Y X=[X,Y]$)

From $[X,Y]^{\mu}=X^{\lambda}\partial_{\lambda}Y^{\mu}-Y^{\lambda}\partial_{\lambda}X^{\mu}=X^{\lambda}\nabla_{\lambda}Y^{\mu}-Y^{\lambda}\nabla_{\lambda}X^{\mu}$ ，we have

$S^{\rho}\nabla_{\rho}T^{\mu}=T^{\rho}\nabla_{\rho}S^{\mu}$ this means that $\nabla_S T=\nabla_T S$

$$
\begin{aligned}
A^{\mu} &= T^{\rho}\nabla_{\rho}(T^{\sigma}\nabla_{\sigma}S^{\mu}) \\
&= T^{\rho}\nabla_{\rho}(S^{\sigma}\nabla_{\sigma}T^{\mu}) \\
&= (T^{\rho}\nabla_{\rho}S^{\sigma})(\nabla_{\sigma}T^{\mu}) + T^{\rho}S^{\sigma}\nabla_{\rho}\nabla_{\sigma}T^{\mu} \\
&= (S^{\rho}\nabla_{\rho}T^{\sigma})(\nabla_{\sigma}T^{\mu}) + T^{\rho}S^{\sigma}(\nabla_{\sigma}\nabla_{\rho}T^{\mu} + R^{\mu}{}_{\nu\rho\sigma}T^{\nu}) \\
&= (S^{\rho}\nabla_{\rho}T^{\sigma})(\nabla_{\sigma}T^{\mu}) + S^{\sigma}\nabla_{\sigma}(T^{\rho}\nabla_{\rho}T^{\mu}) - (S^{\sigma}\nabla_{\sigma}T^{\rho})\nabla_{\rho}T^{\mu} \\
&\quad + R^{\mu}{}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma} \\
&= R^{\mu}{}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma}. \qquad (3.207)
\end{aligned}
$$

$A^{\mu}=T^{\rho}\nabla_{\rho}(T^{\sigma}\nabla_{\sigma}S^{\mu})$ Relative acceleraion of geodesics $=\nabla_T(\nabla_T S)$

$=T^{\rho}\nabla_{\rho}(S^{\sigma}\nabla_{\sigma}T^{\mu})$ $\because \nabla_T S=\nabla_S T$

$=(T^{\rho}\nabla_{\rho}S^{\sigma})(\nabla_{\sigma}T^{\mu})+T^{\rho}S^{\sigma}\nabla_{\rho}\nabla_{\sigma}$ Leibniz rule

$=(S^{\rho}\nabla_{\rho}T^{\sigma})(\nabla_{\sigma}T^{\mu})+T^{\rho}S^{\sigma}(\nabla_{\sigma}\nabla_{\rho}T^{\mu}+R_{\nu\rho\sigma}$

$=(S\nabla_{\rho}T\nabla(\ ^{\mu}T)\nabla S\nabla_{\rho}T^{\mu}T\nabla)_{\sigma}-(S\nabla_{\rho}T^{\mu})_{\nu\rho\sigma}T$

$=R^{\mu}_{\nu\rho\sigma}T^{\nu}T^{\rho}$

$A^{\mu}=\frac{D^2}{dt^2}S^{\mu}=R^{\mu}_{\nu\rho\sigma}T^{\nu}T^{\rho}S^{\sigma}$ is the geodesic deviation equation。

By a Jacobi field usually is meant a vector field along a geodesic in a pseudo-Riemannian manifold which obeys the geodesic deviation equation。

設 J 為測地線 $\gamma(t)$ 上的向量場，若

$\nabla_T^2 J + R(J,T)T = 0$，其中 $T = \frac{d\gamma}{dt}$，($\nabla_T^2 J$ 即 $\frac{D^2 J}{dt^2}$)

則稱 J 為 $\gamma(t)$ 上的 Jacobi 場。

定理 J 是 γ 上切於 geodesic variation $\hat{\gamma}$ 上 $\Leftrightarrow$ J 是 γ 上的一個 Jacobi 場。

$\Rightarrow$

$$R(J,T)T = \nabla_J \nabla_T T - \nabla_T \nabla_J T - \nabla_{[J,T]} T$$

其中 $\nabla_T T = 0$，$\nabla_J T = \nabla_T J$，$[J,T] = 0$ ($\because [\frac{\partial \hat{\gamma}}{\partial v}, \frac{\partial \hat{\gamma}}{\partial t}] = 0$)

所以 $\nabla_T^2 J + R(J,T)T = 0$