Physikalisches Institut
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Theoretische Physik

Exercise Sheet 8
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SS 17

Exercises General Relativity and Cosmology

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–Homework–

1 Killing vectors (17 points)

A principle in General Relativity is to have invariance under general coordinate redefinitions. This fact is often called *diffeomorphism invariance.* This exercise studies diffeomorphisms that leave the metric unchanged, so-called *isometries*: A diffeomorphism $f : M \to M$ is an isometriy if it preserves the metric, i.e.

$$f^* g_{f(p)} = g_p \,, \tag{1}$$

that is if $g_{f(p)}(f_* X, f_* Y) = g_p(X, Y)$ for $X, Y \in T_p M$, or in components

$$\frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(f(p)) = g_{\mu\nu}(p) \,, \tag{2}$$

where x and y are the coordinates of p and $f(p)$ respectively. Isometries naturally form a group which we are going to study from an infinitesimal point of view.

a) A vector field $K = K^\mu \partial_\mu$ on M is said to be a *Killing vector field* if the infinitesimal displacement $f : x^\mu \mapsto x^\mu + \epsilon K^\mu$ generates an isometry. Show that this is the case, if (*1 point*)

$$X^\kappa \partial_\kappa g_{\mu\nu} + \partial_\mu X^\kappa g_{\kappa\nu} + \partial_\nu X^\kappa g_{\mu\kappa} = 0 \,. \tag{3}$$

These are the so-called *Killing equations*.

b) Show that the Killing equations can be written as (*1 point*)

$$g\left(\nabla_X K, Y\right) + g\left(X, \nabla_Y K\right) = 0, \tag{4}$$

for any vector fields X and Y.

Here, we consider the Killing vector fields on the two sphere $M = S^2$. These are given by

$$\begin{aligned} K_1 &= -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi \ , \\ K_2 &= +\cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi \ , \\ K_3 &= \partial_\phi \ . \end{aligned} \tag{5}$$

Here, θ and ϕ are local coordinates of S^2, in terms of which the metric reads

$$g = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 \ . \tag{6}$$

c) Verify that the set of Killing vector fields (5) satisfy the Killing equations (3) for the metric (6). (*1 point*)

As we will see below, Killing vector fields can be used to find the geodesics on M.

d) Consider a geodesic $\mathcal{C}$ parameterized by λ, where λ is affinely related to the arc length. Deduce from the defintion (4) that for a Killing vector field K

$$g\left(K, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) = \text{constant along the geodesic } \mathcal{C} \; . \tag{7}$$

These are first order differential equations for the geodesic. (*2 points*)

e) On the exercise sheet 4, we have introduced the Lie bracket (commutator) of vector fields, denoted as $[X, Y]$. Show that

$$[K_i, K_j] = -\epsilon_{ijk} K_k \;, \quad i, j, k = 1, 2, 3 \;, \tag{8}$$

with the totally antisymmetric ϵ-tensor that is normalized by $\epsilon_{123} = 1$. (*3 points*)

f) Note that the vector fields $-iK_j$ for $j = 1, 2, 3$ and $i^2 = -1$ fulfill the angular momentum algebra. Can you think of a reason for this? (*2 points*)

g) For the Killing vector fields given in eq. (5), write eqs. (7) in terms of the local coordinates. Label the constant appearing on the right hand side of (7) for $K = K_i$ as L_i (*3 points*)

h) Combine the equations found in the previous item to arrive at an equation, in which θ, ϕ and the L_i but no derivatives of the geodesic appear. This equation can (at least locally) be solved to yield the geodesics in the form $\theta(\phi)$ or $\phi(\theta)$. (*2 points*)

i) Find the geodesics for the three cases in which only one L_i is non-zero. (*2 points*)

Killing vector fields on a manifold are in one-to-one correspondence with continuous symmetries of the metric on the manifold. Every Killing vector implies the existence of conserved quantities associated with geodesic motion. Physically this can be understood in the following way: A particle moving along the direction of the Killing vector will not feel forces and the component of momentum in that direction will be conserved.

The number of isometries is given for a number of linearly independent Killing vector fields, i.e. the number of independent solutions of the Killing equations (3). We refer to an n-dimensional manifold with $\frac{1}{2}n(n+1)$ Killing vectors as a *maximally symmetric space*. This happens to be the case for Euclidean spaces $\mathbb{R}^n$ and n-spheres S^n. Other maximally symmetric spaces of interest in cosmology are the de Sitter $\mathrm{d}S_n$ and Anti-de Sitter $\mathrm{Ad}S_n$ geometries, which are Lorentzian manifolds with positive and negative curvature respectively.

2 Relativistic and gravitational redshift (15 points)

One of the striking predicitions of general relativity is the gravitational redshift of light, which has been experimentally verified by the *Pound-Rebka* experiment. In this exercise we elaborate on what has been said about this experiment in the lecture at some point of spacetime.

As usual, we set $c = \hbar = 1$. Consider an observer O with four-velocity U and a photon moving along a geodesic $\mathcal{C}$, which is parameterized such that its tangent vector $\mathrm{d}\mathcal{C}/\mathrm{d}\lambda$ equals the photon's four-momentum P. Then the observer O measures the photon's angular frequency as

$$\omega_O = -g\left(U, \frac{\mathrm{d}\mathcal{C}}{\mathrm{d}\lambda}\right) , \tag{9}$$

where g is the space-time metric at the point of measurement.

a) Find the Killing vector K_t of the Schwarzschild metric, which leads you to the following differential equation *(2 points)*

$$\left(1 - \frac{R_S}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda} = E, \tag{10}$$

where $R_S = 2GM$. Here, λ is the same as the one in and above eq. (9). Further, E is a constant, namely the energy (frequency) that an observer resting at infinity would assign to the photon.

b) Let us first consider the relativistic Doppler effect, which is already present in special relativity, i.e. we take $g = \eta$ with the Minkowski metric η. Write down the four-velocity U_1 of an observer O_1 in its rest-frame. In this coordinate system, further write down the four-momentum P of a photon that travels along the unit direction $\vec{e}_\gamma$ and whose angular frequency is by O_1 measured to be ω_1. *(3 points)*

c) In the same coordinate system, write down the four-velocity U_2 of a second observer O_2, who by O_1 is measured to travel with velocity $\vec{v} = v\vec{e}_2$, $|\vec{v}| = v$. Show that O_2 measures the photon's angular frequency as *(3 points)*

$$\omega_2 = \gamma(v)\omega_1\left(1 - v\vec{e}_\gamma \cdot \vec{e}_2\right) . \tag{11}$$

d) Now we consider to the gravitational redshift in the Schwarzschild metric. Look up this metric in your lecture notes and write down the four-veloctiy of an observer that is static in the Schwarzschild coordinates and sits at radius r. Consider two of these observers, one at $r = r_1$, the other one at $r = r_2$. Denote the frequencies they measure as $\omega(r_1)$ and $\omega(r_2)$ respectively and find an expression for the ratio $\omega(r_1)/\omega(r_2)$. *(2 points)*

e) We are interested in the following situation: The Schwarzschild metric is used as an approximation for the metric outside the surface of earth. Further, we take $r_2 = R_0$ as the radius of Earth and set $r_1 = r_2 + h$ with the height h of a tall building. Use this to appropriately approximate your result from item c). You should find

$$\omega(r_1)/\omega(r_2) = 1 - g \cdot h , \tag{12}$$

where g is the gravitational acceleration on the surface of Earth. *(2 points)*

f) In the Pound-Rebka experiment, a photon emitting ^{57}Fe source was mounted on a loudspeaker at the top of a tower. Both the photon emission and the movement of the source due to turning on the loudspeaker is along the tower (perpendicular to the surface of Earth). At the bottom of the tower, a ^{57}Fe source is mounted above a detector. By looking for the minimum of the counting rate, the velocity of the source at which the relativistic Doppler effect cancels the gravitational redshift has been measured. Find the velocity (both the absolute value as well as the direction) at which this happens to first order in v and h. *(3 points)*