**1.** A space monkey is happily orbiting a Schwarzschild black hole in a circular geodesic orbit. An evil baboon, far from the black hole, tries to send the monkey to its death inside the black hole by dropping a carefully timed coconut radially toward the black hole, knowing that the monkey can't resist catching the falling coconut. Given the monkey's mass and initial orbital radius and the mass of the coconut, explain how you would go about solving the problem (but do not do the calculation). What are the possible fates for our intrepid space monkey?

**2.** Consider a perfect fluid in a static, circularly symmetric $(2+1)$-dimensional spacetime, equivalently, a cylindrical configuration in $(3+1)$ dimensions with perfect rotational symmetry.

**(a)** Derive the analogue of the Tolman–Oppenhiemer–Volkov (TOV) equation for $(2+1)$ dimensions.

**(b)** Show that the vacuum solution can be written as

$$ds^2 = -dt^2 + \frac{1}{1-8GM}dr^2 + r^2 d\theta^2$$

Here $M$ is a constant.

**(c)** Show that another way to write the same solution is

$$ds^2 = -d\tau^2 + d\xi^2 + \xi^2 d\phi^2$$

where $\phi \in [0, 2\pi(1-8GM)^{1/2}]$.

**(d)** Solve the $(2+1)$ TOV equation for a constant density star. Find $p(r)$ and solve for the metric.

**(e)** Solve the $(2+1)$ TOV equation for a star with equation of state $p = \kappa\rho^{3/2}$. Find $p(r)$ and solve for the metric.

**(f)** Find the mass $M(R) = \int_0^{2\pi}\int_0^R \rho\, dr\, d\theta$ and the proper mass $\bar{M}(R) = \int_0^{2\pi}\int_0^R \rho\sqrt{-g}\, dr\, d\theta$ for the solutions in parts (d) and (e).

**3.** Consider a particle (not necessarily on a geodesic) that has fallen inside the event horizon, $r < 2GM$. Use the ordinary Schwarzschild coordinates $(t, r, \theta, \phi)$. Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. Express this in seconds for a black hole with mass measured in solar masses. Show that this maximum proper time is achieved by falling freely with $E \to 0$.

**4.** Consider Einstein's equations in vacuum, but with a cosmological constant, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$.

**(a)** Solve for the most general spherically symmetric metric, in coordinates $(t, r)$ that reduce to the ordinary Schwarzschild coordinates when $\Lambda = 0$.

**(b)** Write down the equation of motion for radial geodesics in terms of an effective potential, as in (5.66). Sketch the effective potential for massive particles.

**5.** Consider a comoving observer sitting at constant spatial coordinates $(r_*, \theta_*, \phi_*)$, around a Schwarzschild black hole of mass $M$. The observer drops a beacon into the black hole (straight down, along a radial trajectory). The beacon emits radiation at a constant wavelength $\lambda_{\rm em}$ (in the beacon rest frame).

**(a)** Calculate the coordinate speed $dr/dt$ of the beacon, as a function of $r$.

**(b)** Calculate the proper speed of the beacon. That is, imagine there is a comoving observer at fixed $r$, with a locally inertial coordinate system set up as the beacon passes by, and calculate the speed as measured by the comoving observer. What is it at $r = 2GM$?

**(c)** Calculate the wavelength $\lambda_{\rm obs}$, measured by the observer at $r_*$, as a function of the radius $r_{\rm em}$ at which the radiation was emitted.

**(d)** Calculate the time $t_{\rm obs}$ at which a beam emitted by the beacon at radius $r_{\rm em}$ will be observed at $r_*$.

**(e)** Show that at late times, the redshift grows exponentially: $\lambda_{\rm obs}/\lambda_{\rm em} \propto e^{t_{\rm obs}/T}$. Give an expression for the time constant $T$ in terms of the black hole mass $M$.