1. The Lagrange density for electromagnetism in curved space is

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu\right), \tag{4.156}$$

where J^μ is the conserved current.

(a) Derive the energy-momentum tensor by functional differentiation with respect to the metric.

(b) Consider adding a new term to the Lagrangian,

$$\mathcal{L}' = \beta R^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}.$$

How are Maxwell's equations altered in the presence of this term? Einstein's equation? Is the current still conserved?

2. We showed how to derive Einstein's equation by varying the Hilbert action with respect to the metric. They can also be derived by treating the metric and connection as independent degrees of freedom and varying separately with respect to them; this is known as the **Palatini formalism**. That is, we consider the action

$$S = \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma),$$

where the Ricci tensor is thought of as constructed purely from the connection, not using the metric. Variation with respect to the metric gives the usual Einstein's equations, but for a Ricci tensor constructed from a connection that has no a priori relationship to the metric. Imagining from the start that the connection is symmetric (torsion free), show that variation of this action with respect to the connection coefficients leads to the requirement that the connection be metric compatible, that is, the Christoffel connection. Remember that Stokes's theorem, relating the integral of the covariant divergence of a vector to an integral of the vector over the boundary, does not work for a general covariant derivative. The best strategy is to write the connection coefficients as a sum of the Christoffel symbols $\widetilde{\Gamma}^\lambda_{\mu\nu}$ and a tensor $C^\lambda{}_{\mu\nu}$,

$$\Gamma^\lambda_{\mu\nu} = \widetilde{\Gamma}^\lambda_{\mu\nu} + C^\lambda{}_{\mu\nu},$$

and then show that $C^\lambda{}_{\mu\nu}$ must vanish.

3. The four-dimensional δ-function on a manifold M is defined by

$$\int_M F(x^\mu)\left[\frac{\delta^{(4)}(x^\sigma - y^\sigma)}{\sqrt{-g}}\right]\sqrt{-g}\, d^4x = F(y^\sigma), \tag{4.157}$$

for an arbitrary function $F(x^\mu)$. Meanwhile, the energy-momentum tensor for a pressureless perfect fluid (dust) is

$$T^{\mu\nu} = \rho U^\mu U^\nu, \tag{4.158}$$

where ρ is the energy density and U^μ is the four-velocity. Consider such a fluid that consists of a single particle traveling on a world line $x^\mu(\tau)$, with τ the proper time. The energy-momentum tensor for this fluid is then given by

$$T^{\mu\nu}(y^\sigma) = m \int_M \left[\frac{\delta^{(4)}(y^\sigma - x^\sigma(\tau))}{\sqrt{-g}}\right]\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau, \tag{4.159}$$

where m is the rest mass of the particle. Show that covariant conservation of the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, implies that $x^\mu(\tau)$ satisfies the geodesic equation.

4. Show that the energy-momentum tensors for electromagnetism and for scalar field theory satisfy the dominant energy condition, and thus also the weak, null, and null dominant conditions. Show that they also satisfy $w \geq -1$.

5. A spacetime is static if there is a timelike Killing vector that is orthogonal to spacelike hypersurfaces. (See the Appendices for more discussion, including a definition of Raychaudhuri's equation.)

(a) Generally speaking, if a vector field v^μ is orthogonal to a set of hypersurfaces defined by $f =$ constant, then we can write the vector as $v_\mu = h\nabla_\mu f$ (here both f and h are functions). Show that this implies

$$v_{[\sigma}\nabla_\mu v_{\nu]} = 0.$$

(b) Imagine we have a perfect fluid with zero pressure (dust), which generates a solution to Einstein's equations. Show that the metric can be static only if the fluid four-velocity is parallel to the timelike (and hypersurface-orthogonal) Killing vector.

(c) Use Raychaudhuri's equation to prove that there is no static solution to Einstein's equations if the pressure is zero and the energy density is greater than zero.

6. Let K be a Killing vector field. Show that an electromagnetic field with potential $A_\mu = K_\mu$ solves Maxwell's equations if the metric is a vacuum solution to Einstein's equations. This is a slight cheat, since you won't be in vacuum if there is a nonzero electromagnetic field strength, but we assume the field strength is small enough not to dramatically affect the geometry.