1. Just because a manifold is topologically nontrivial doesn't necessarily mean it can't be covered with a single chart. In contrast to the circle S^1, show that the infinite cylinder $\mathbf{R} \times S^1$ can be covered with just one chart, by explicitly constructing the map.

2. By clever choice of coordinate charts, can we make $\mathbf{R}^2$ look like a one-dimensional manifold? Can we make $\mathbf{R}^1$ look like a two-dimensional manifold? If so, explicitly construct an appropriate atlas, and if not, explain why not. The point of this problem

   is to provoke you to think deeply about what a manifold is; it can't be answered rigorously without going into more details about topological spaces. In particular, you might have to forget that you already know a definition of "open set" in the original $\mathbf{R}^2$ or $\mathbf{R}^1$, and define them as being appropriately inherited from the $\mathbf{R}^1$ or $\mathbf{R}^2$ to which they are being mapped.

3. Show that the two-dimensional torus T^2 is a manifold, by explicitly constructing an appropriate atlas. (Not a maximal one, obviously.)

4. Verify the claims made about the commutator of two vector fields at the end of Section 2.3 (linearity, Leibniz, component formula, transformation as a vector field).

5. Give an example of two linearly independent, nowhere-vanishing vector fields in $\mathbf{R}^2$ whose commutator does not vanish. Notice that these fields provide a basis for the tangent space at each point, but it cannot be a coordinate basis since the commutator doesn't vanish.

6. Consider $\mathbf{R}^3$ as a manifold with the flat Euclidean metric, and coordinates $\{x, y, z\}$. Introduce spherical polar coordinates $\{r, \theta, \phi\}$ related to $\{x, y, z\}$ by

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta, \end{aligned} \tag{2.99}$$

   so that the metric takes the form

$$ds^2 = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\phi^2. \tag{2.100}$$

(a) A particle moves along a parameterized curve given by

$$x(\lambda) = \cos\lambda, \quad y(\lambda) = \sin\lambda, \quad z(\lambda) = \lambda. \tag{2.101}$$

Express the path of the curve in the $\{r, \theta, \phi\}$ system.

(b) Calculate the components of the tangent vector to the curve in both the Cartesian and spherical polar coordinate systems.

7. Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual cartesian coordinates (x, y, z) of Euclidean three-space by

$$x = \sinh\chi \, \sin\theta \, \cos\phi,$$

$$y = \sinh\chi \, \sin\theta \, \sin\phi,$$

$$z = \cosh\chi \, \cos\theta.$$

Restrict your attention to the plane $y = 0$ and answer the following questions.

(a) What is the coordinate transformation matrix $\partial x^{\mu}/\partial x^{\nu'}$ relating (x, z) to (χ, θ)?

(b) What does the line element ds^2 look like in prolate spheroidal coordinates?

8. Verify (2.78): for the exterior derivative of a product of a p-form ω and a q-form η, we have

$$\mathrm{d}(\omega \wedge \eta) = (\mathrm{d}\omega) \wedge \eta + (-1)^p \omega \wedge (\mathrm{d}\eta). \tag{2.102}$$

9. In Euclidean three-space, suppose $*F = q \sin\theta \, \mathrm{d}\theta \wedge \mathrm{d}\phi$.

(a) Evaluate $\mathrm{d} * F = *J$.

(b) What is the two-form F equal to?

(c) What are the electric and magnetic fields equal to for this solution?

(d) Evaluate $\int_V \mathrm{d}{*F}$, where V is a ball of radius R in Euclidean three space.

10. Consider Maxwell's equations, $\mathrm{d}F = 0$, $\mathrm{d}{*F} = *J$, in 2-dimensional spacetime. Explain why one of the two sets of equations can be discarded. Show that the electromagnetic field can be expressed in terms of a scalar field. Write out the field equations for this scalar field in component form.

11. There are a lot of motivational words attached here to what is a very simple problem; don't get too distracted. In ordinary electromagnetism with point particles, the part of the action which represents the coupling of the gauge-potential one-form $A^{(1)}$ to a charged particle can be written $S = \int_\gamma A^{(1)}$, where γ is the particle worldline. (The superscript on $A^{(1)}$ is just to remind you that it is a one-form.) For this problem you will consider a theory related to ordinary electromagnetism, but this time in 11 spacetime dimensions, with a three-form gauge potential $A^{(3)}$ and four-form field strength $F^{(4)} = \mathrm{d}A^{(3)}$. Note that the field strength is invariant under a gauge transformation $A^{(3)} \to A^{(3)} + \mathrm{d}\lambda^{(2)}$ for any two-form $\lambda^{(2)}$.

(a) What would be the number of spatial dimensions of an object to which this gauge field would naturally couple (for example, ordinary E+M couples to zero-dimensional objects—point particles)?

(b) The electric charge of an ordinary electron is given by the integral of the dual of the two-form gauge field strength over a two-sphere surrounding the particle. How would you define the "charge" of the object to which $A^{(3)}$ couples? Argue that it is conserved if $\mathrm{d} * F^{(4)} = 0$.

(c) Imagine there is a "dual gauge potential" $\widetilde{A}$ that satisfies $\mathrm{d}(\widetilde{A}) = *F^{(4)}$. To what dimensionality object does it naturally couple?

(d) The action for the gauge field itself (as opposed to its coupling to other things) will be an integral over the entire 11-dimensional spacetime. What are the terms that would be allowed in such an action that are invariant under "local" gauge transformations, for instance, gauge transformations specified by a two-form $\lambda^{(2)}$ that vanishes at infinity? Restrict yourself to terms of first, second, or third order in $A^{(3)}$ and its first derivatives (no second derivatives, no higher-order terms). You may use the exterior derivative, wedge product, and Hodge dual, but not any explicit appearance of the metric.

More background: "Supersymmetry" is a hypothetical symmetry relating bosons (particles with integral spin) and fermions (particles with spin $\frac{1}{2}$, $\frac{3}{2}$, etc.). An interesting feature is that supersymmetric theories are only well-defined in 11 dimensions or less—in larger numbers of dimensions, supersymmetry would require the existence of particles with spins greater than 2, which cannot be consistently quantized. Eleven-dimensional supersymmetry is a unique theory, which naturally includes a three-form gauge potential (not to mention gravity). Recent work has shown that it also includes the various higher-dimensional objects alluded to in this problem (although we've cut some corners here). This theory turns out to be a well-defined limit of something called M-theory, which has as other limits various 10-dimensional superstring theories.