Spettral Theory in Riemannian geometry　by Olivier Lablee

§ Introduction

R^3 Laplace operator (Laplacian) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(M,g) Laplace-Beltrami operator $\Delta_g = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j)$

(Jacobi operator $J = \Delta + |A|^2 + Ric(N,N)$)

Spectrum 與幾何有密切關係，Δ_g 及其譜的研究稱為 spectral geomery。

(M,g) $-\Delta_g$ ：the operator is self-adjoint and its spetrum is discete。

例

1. Navier-Stokes equation

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u - v\Delta u = -\nabla P \\ div(u) = 0 \end{cases}$$

2. Potential theory and gravity theory

$\Delta u = f$

3. Heat equation

$$\frac{\partial u(x,t)}{\partial t} - \Delta u(x,t) = f(x,t)$$

4. Wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = 0$$

5. Schrodinger equation

§ Problems

1. Direct problems
   (1) Compute Spec(M,g)　the first non-null eigenvalue λ_1
   (2) Properties of Spec(M,g)
2. Inverse problems
   (1) Spec(M,g)決定了 M 的 dimension，volume，the integral of scalar curvature(over M)
   (2) Isospectral problem：Can we hear the shap of a drum？
   If (M,g)，(M',g') are isospectral，are they isometric？ J.Milnor 1964 and a planar counter example by C.Gordon、D.Webb and S.Wolpert 1992

§ Laplacian operator 的幾何意義

設$u: R \to R$ 考慮 u 在[-h,h]的平均值$\bar{u} = \frac{1}{2h}\int_{-h}^{h} u(x)dx$

u(x)在 x=0 展開$u(x) = u(0) + u'(0)x + u''(0)\cdot\frac{x^2}{2} + u'''(0)\cdot\frac{x^3}{6} + o(x^4)$

則$\bar{u} = u(0) + \frac{u''(0)}{6}h^2 + o(h^4)$

$u''(0) = \frac{6}{h^2}(\bar{u} - u(0)) + o(h^2)$ （$\Delta u(x) = \frac{d^2u}{dx^2}$所以 u 的 Laplacian 在度量 u(0)與 u 在其鄰域的平均值的差)