§ 2019 年台大博士班

1. Consider the orthogonal group： $O(n)=\{g\in GL(n;R):g^{-1}=g^{T}\}$
   Show that O(n) is a dfferentiable manifold，and determine its dimension。

2. On R^3，consider the following metric
   $ds^2=dx^2+dy^2+(dz+\sin zdx+\cos zdy)^2$
   (a) Calculate the Riemann curvatre tensor of ds^2
   (b) Denote the 1-form $dz+\sin zdx+\cos zdy$ by α。
   Can you find a regular surface Σ passing through the origin，and
   $T_p\Sigma\subset\ker(\alpha\big|_p)$ for every $p\in\Sigma$? Justify your answer。
   (c) Same question as (b) for the 1-form $\beta=dz+zdx$
   Can you find a regular surface Σ passing the origin，and $T_p\Sigma\subset\ker(\beta\big|_p)$
   for every $p\in\Sigma$? Justify your answer。

3. On a Remannian manifold (M, g)，a vector field U is called a Killing vector field if it is infinitesimally an isometry，namely $\frac{d}{dt}\Big|_{t=0}\varphi_t^{*}g=0$
   Where φ_t is the one-paremeter family of diffeomorphism generate by U
   (a) For a Killing vector field U，show that $g(\nabla_X U,Y)+g(\nabla_Y U,X)=0$ for any two vector fields X,Y
   (b) Suppose that U is a Killing vector field，and γ is a geodesic。Prove that $U\big|_\gamma$ is a Jacobi field

4. Let (M, g) be a connected Riemannian manifold，with $\dim M\geq 3$
   (a) State the second Bianchi identity for the Riemann crvature
   (b) Sippose that its Ricci curvature is proportional to the metric tensor。Namely，thre exists $f\in C^{\infty}(M;R)$ such that Ric(X ,Y)=f(p)g(X, Y) for any $p\in M$，and $X,Y\in T_pM$。Prove that f must be a contstant function。

   Hint : In terms of coordinate，the condition reads $R^{l}_{ilj}=fg_{ij}$。

   Taking covariant derivative in ∂_k gives $R^{l}_{ilj;k}=(\partial_k f)g_{ij}$

5. Let Σ be a closed (compact without boundary)、oriented surface。
   (a) Does there always exist a smooth map，$F:\Sigma \to S^2$ such that F is essential (i.e. F is not homotopic to a constant map)？Justify your answer。
   (b) Consider the same question，but replacing the codomain S^2 by the 2-torus T^2

1. Let U be a connected open subset of R^2。
   Let f be a smooth function on U and let the metric $ds^2 = e^{2f}(dx^2 + dy^2)$
   (a) Determine the scalar curvature R of the metric
   (b) Use the resulting formula above to find the scalar curvature of the upper half plane with the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$
2. Let M be a n-dimensional Riemannian manifold
   (a) Let $\{e_1, e_2, ..., e_n\}$ be a local frame for the tangent bundle。
   Let ∇ be the Levi-Civita connection。Determine constants {a,b,c}，so that
   $$g(\nabla_{e_i} e_j, e_k) = ag([e_i, e_j], e_k) + bg([e_j, e_k], e_i) + cg([e_k, e_i], e_j)$$
   (b) Show that there exists a local orthonormal fram field with $[e_i, e_j] = 0$ for all i,j if and only if the curvature tensor vanishes identically。
3. A derivation of $C^\infty(R^n)$ based at a point P is a linear map L from $C^\infty(R^n)$ to R satisfying the Leibnitz rule L(fg)=f(P)L(g)+L(f)g(P)。
   Let L be a derivation of $C^\infty(R^n)$ based at P=0，and let $x = (x^1, ..., x^n)$ be the coordinate functions on R^n。
   Show there exist real constants $a_1, ..., a_n$ so that
   $$L(f) = a_1 \frac{\partial f}{\partial x^1}(0) + ... + a_n \frac{\partial f}{\partial x^n}(0)$$
4. Consider (R^2, g) to be the Riemannian manifold，with metric given by
   $$g = (e^{-x} + y^2 e^x)dx^2 + xye^{-\frac{x}{2}}dxdy + 10(x^4 + y^4 + 5)dy^2$$
   (a) Argue that this is a Riemannian metric
   (b) Is this a complete manifold？Prove or give a reason why it would not be。

5. Suppose that a Riemannian manifold has section curvatures of both +1 and 1 at a point p : Prove there exist a (2-dimensional) tangent plane at p that has zero sectional curvature。

6. Let $X = \{(x, y, z) \in R^3 \,|\, x^3 + xyz + y^2 = 1\}$

   (a) Show that X is a 2-manifold

   (b) Consider the map $\pi : X \to R^2$ taking (x,y,z) to (x,y)
   Find all points of X at which π fails to be a local diffeomorphism