§ 2014 年

1.

(a) Let M be a closed differentiable manifold of dimension n。
Show that M can be embedded into R^n for some positive integer n

(b) show that the projective plane RP^2 can be embedded into R^4

2. let (Σ, g) be a smooth Riemann surface with rotationally symmetric metric $g = dr^2 + f^2(r)d\theta^2$ for some positive smooth function f(r)，r>0

(a) compute $K = -\frac{f''}{f}$，where K is the Gaussian Curvature。

(b) in particular，$K = \begin{cases} 1, f(r) = \sin r \\ 0, f(r) = r \\ -1, f(r) = \sinh r \end{cases}$

3. let (M,g) be a complete Riemannian manifold of dimension n with the Ricci curvature bounded from below by (n-1)K，where K>0

(a) sow that $diam(M) \le \frac{\pi}{\sqrt{K}}$

(b) show that M is compact and $\pi_1(M) < \infty$

4. let $f : \Sigma \to R^3$ be an isometric immersion of a smooth closed orientable Riemann surface Σ into R^3。

We define the Willmore energy $W(f) = \int_\Sigma H^2 dA$

Where $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is the mean curvature and κ_1, κ_2 are principal curvature。

(a) Show that $\int_\Sigma K^+ dA \ge 4\pi$，where $K^+ = \max\{K, 0\}$，$\Sigma^+ = \Sigma\big|_{K^+}$ and $K = \kappa_1 \times \kappa_2$ is the Gauss curvature

(b) Show that $W(f) \ge 4\pi$
Moreover，$W(f) = 4\pi$ if and only if Σ is embedded as a round sphere in R^3。i.e. $\kappa_1 = \kappa_2$ at every point。

5.

(a) Let (M,g) be a closed，orientable Riemannian manifold of dimension n。

Show that all de Rham cohomology groups $H_{dR}^{p}(M,R), 0 \leq p \leq n$ 。Are finite dimensional。

(b) Let $\omega_1,...,\omega_n \in R^n$ be linearly independent。

We consder z_1, z_2 in R^n as equivalent if there are $m_1,...,m_n \in Z$ with

$z_1 - z_2 = \sum_{i=1}^{n} m_i \omega_i$ 。

Let π be the projection mapping z to its equivalence class。

Now we consider the n-dimensional torus $T^n := \pi(R^n)$

Compute all p-th Betti number $b_p(T^n)$ for $0 \leq p \leq n$