§ 2009 年 台大博士班考題

1. Let ∇ be the Levi-Civita connection on a Riemannian n-manifold M with metric g_{ij} defined by $g_{ij}=\left\langle \frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right\rangle$ ，I,j=1,…,n

   Defined the Christoffel symbol Γ_{ij}^k by $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}=\Gamma_{ij}^k\frac{\partial}{\partial x^k}$ ，I,j,k=1,…,n

   and the Riemannian curvature tensor R_{ijl}^m by

   $R_m(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}=R_{ijl}^m\frac{\partial}{\partial x^m}$ and $R_{ijkl}=g_{mk}R_{ijl}^m$

   Here $R_m(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j})\frac{\partial}{\partial x^l}=\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^l}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^l}$

   Finally we define the Ricci curvature tensor and scalar curvature by

   $R_{ij}=g^{kl}R_{ikjl}$ and $R=g^{ij}R_{ij}$

   Show that

   (a) $\Gamma_{ij}^k=\Gamma_{ji}^k$

   (b) $\Gamma_{ij}^k=\frac{1}{2}g^{kl}(g_{lj,i}+g_{il,j}-g_{ij,l})$ Here $g_{ij,k}=\frac{\partial g_{ij}}{\partial x^k}$

   (c) $R_{ijl}^m=\frac{\partial}{\partial x^i}\Gamma_{jl}^m-\frac{\partial}{\partial x^j}\Gamma_{il}^m+\Gamma_{in}^m\Gamma_{jl}^n-\Gamma_{jn}^m\Gamma_{il}^n$

   (d) $\nabla_{\frac{\partial}{\partial x^i}}g_{jk}=0$ for all I,j,k

   (e) $\nabla_{\frac{\partial}{\partial x^i}}R_j^i=\frac{1}{2}\nabla_{\frac{\partial}{\partial x^j}}R$

   (f) Suppose that for some smooth function ρ ，we have $R_{ij}=\rho g_{ij}$ on the whole manifold M。Show that ρ is constant and $\rho=\frac{R}{n}$ ，n>2

2. Let $(R^2, g(t))$ be a complete Riemannian surface with $g(t) = \dfrac{dx^2 + dy^2}{e^{4t} + x^2 + y^2}$

   $dx^2 = dx \otimes dx$ 。Show taht

   (a) In polar coordinates (r, θ) ，we may rewrite

   $$g(0) = ds^2 + \tanh^2 s d\theta^2, s = \log(r + \sqrt{1 + r^2})$$

   (b) The scalar curvature of $(R^2, g(0))$ ，$R_0 = \dfrac{4}{1 + r^2}$

   (c) Find 1-parameter group of conformal diffeomorphisms $\varphi_t : R^2 \to R^2$

   Such that $g(t) = \varphi_t^* g(0)$

3. Let M be an oriented differentiable n-manifold and $H_{deR}^p(M, R)$ be the pth de Rham cohomology group。Show that

   (a) If M is a closed manifold，then $\dim(H_{deR}^p(M, R)) < \infty$

   (b) If $M = R^n$ ，then $H_{deR}^p(M, R) = 0$ for all p>0