§ 2006 年台大博士班考題

1. Let $\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$ be the Cartesian coordinates of the Euclidean space

R^3，and $\eta = \frac{1}{\rho^3}(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)$ be a differential 2-form on the

open set $D = \{(x,y,z) \neq (0,0,0)\}$。

Is η a closed 2-form？$\eta = d\omega$？If yes，$\omega = ?$

If you cannot find an ω on the whole domain D，let us restrict ourselves to

$D^+ = \{(x,y,z) \in \mathrm{D} | x > 0\}$。

Can toy find an anti-derivative ω in this smaller domain D^+？

2. $S^3 = \{(x,y,z) \in R^4 | x^2 + y^2 + z^2 + w^2 = 1\}$ is a 3-dimensional manifold with

positive Ricci curvature。

Can you find a different Riemannian metric on S^3 so that at the point (x,y,z,w)=(0,0,0,1) the Ricci curvature is not non-negative $Ric(0,0,0,1) < 0$，yet the Ricci curvature at (0,0,0,1) is not no-positive either，Ric(0,0,0,1)>0

2. $S^3 = \{(x,y,z) \in \mathbb{R}^4 | x^2 + y^2 + z^2 + w^2 = 1\}$ is a 3-dimensional manifold with positive Ricci curvature. Can you find a different riemannian metric on S^3 so that at the point $(x,y,z,w) = (0,0,0,1)$ the Ricci curvature is not non-negative $Ric(0,0,0,1) \not\geq 0$, yet the Ricci curvature at $(0,0,0,1)$ is not no-positive either, $Ric(0,0,0,1) \not\leq 0$? (25/100)

3. Let $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$ be the Poincare metric on the upper half plane

$H = \{(x,y) | y > 0\}$，and $\gamma = \{x^2 + y^2 = 2 | -1 \leq x \leq 1\}$ be an arc from p(-1,1) to

q(1,1)。Let $\vec{u} = (1,0)$ be a vector at p and $\vec{v}$ be the parallel translation of $\vec{u}$ from

p to q along γ。$\vec{v} = (?,?)$

4. Mean curvature is not an intrinsic quantity of suefaces in R^3 。
Can you find an isometry map from the helocoid
$H = \left\{ \tan z = \frac{y}{x} \middle| x^2 + y^2 < 1, 0 < z < \pi \right\}$ to another surface ，which is not congruent to H，ye preserving the mean curvature of H？