§ 2004 年 台大博士班考題

1. Let $X=\sum_{i=1,2} a_i(x_1,x_2)\frac{\partial}{\partial x^i}, Y=\sum_{i=1,2} b_i(x_1,x_2)\frac{\partial}{\partial x^i}$ be vector fields on R^2。

   (a) Find a formula for the bracket [X,Y]

   (b) Let ω be a differential p-form on an n-dimensional differentiable manifold M，and d be the exterior derivative on M。
   Find the formula for $d\omega$ by using local coordinates，and show that your answer is independent of the choice of coordinates。

2. Let M be a simply connected n-dimensional differentiable manifold。
   Let ω be differential 1-form。Suppose that ω is closed ，i.e. $d\omega = 0$。
   Show that ω is exact ，i.e. there exists a differentiable function f on M such that $df = \omega$

3. Let T be a topological torus，i.e. diffeomorphic to R^2/Z^2。
   Let g be a Riemannian metric on T，written as
   $g = a(x,y)dx^2 + 2b(x,y)dxdy + c(x,y)dy^2$ where (x,y) are the Euclidean coordinates on R^2。
   Suppose the Gaussian curvature $K(g) \le 0$ everywhere。
   Find all possible solutions of g。

4. Prove or disprove that the tangent bundle $T(S^2)$ of the 2-dimensional sphere S^2 is a topologically non-trivial vector bundle，i.e. $T(S^2)$ is not equivalent to the trivial bundle $R^2 \times S^2$ over S^2