§ Riemannian Geometry Peter Petersen

Chapter 1 Riemannian Metrics

(1) On product manifolds $M \times N$ one has special product metrics $g = g_M + g_N$, where g_M, g_N are metrics on M, N respectively.

(a) Show that $(\mathbb{R}^n, \mathrm{can}) = \left(\mathbb{R}, dt^2\right) \times \cdots \times \left(\mathbb{R}, dt^2\right)$.

(b) Show that the flat square torus

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2 = \left(S^1, \left(\frac{1}{2\pi}\right)^2 d\theta^2\right) \times \left(S^1, \left(\frac{1}{2\pi}\right)^2 d\theta^2\right).$$

(c) Show that

$$F(\theta_1, \theta_2) = \frac{1}{2}(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)$$

is a Riemannian embedding: $T^2 \to \mathbb{R}^4$.

(2) Suppose we have an isometric group action G on (M, g) such that the quotient space M/G is a manifold and the quotient map a submersion. Show that there is a unique Riemannian metric on the quotient making the quotient map a Riemannian submersion.

(3) Construct paper models of the Riemannian manifolds $\left(\mathbb{R}^2, dt^2 + a^2t^2d\theta^2\right)$. If $a = 1$, this is of course the Euclidean plane, and when $a < 1$, they look like cones. What do they look like when $a > 1$?

(4) Suppose φ and ψ are positive on $(0, \infty)$ and consider the Riemannian submersion

$$\left((0,\infty) \times S^3 \times S^1,\ dt^2 + \varphi^2(t)\left[(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right] + \psi^2(t)d\theta^2\right)$$
$$\downarrow$$
$$\left((0,\infty) \times S^3,\ dt^2 + \varphi^2(t)[(\sigma^2)^2 + (\sigma^3)^2] + \tfrac{(\varphi(t)\cdot\psi(t))^2}{\varphi^2(t)+\psi^2(t)}(\sigma^1)^2\right).$$

Define $f = \varphi$ and $h = \frac{(\varphi(t)\cdot\psi(t))^2}{\varphi^2(t)+\psi^2(t)}$ and assume that

$$\begin{aligned} f(0) &> 0, \\ f^{(\mathrm{odd})}(0) &= 0, \end{aligned}$$

and

$$\begin{aligned} h(0) &= 0, \\ h'(0) &= k, \\ h^{(\text{even})}(0) &= 0, \end{aligned}$$

where k is a positive integer. Show that the above construction yields a smooth metric on the vector bundle over S^2 with Euler number $\pm k$. Hint: Away from the zero section this vector bundle is $(0,\infty) \times S^3/\mathbb{Z}_k$, where $S^3/\mathbb{Z}_k$ is the quotient of S^3 by the cyclic group of order k acting on the Hopf fiber. You should use the submersion description and then realize this vector bundle as a submersion of $S^3 \times \mathbb{R}^2$. When $k = 2$, this becomes the tangent bundle to S^2. When $k = 1$, it looks like $\mathbb{C}P^2 - \{\text{point}\}$.

(5) Let G be a compact Lie group
   (a) Show that G admits a bi-invariant metric, i.e., both right and left translations are isometries. Hint: Fix a left invariant metric g_L and a volume form $\omega = \sigma^1 \wedge \cdots \wedge \sigma^1$ where σ^i are left invariant 1-forms. Then define g as the average over right translations:

$$g(v,w) = \frac{1}{\int \omega} \int g_L(DR_x(v), DR_x(w))\,\omega.$$

   (b) Show that the *inner automorphism* $\mathrm{Ad}_h(x) = hxh^{-1}$ is a Riemannian isometry. Conclude that its differential at $x = e$ denoted by the same letters

$$\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}$$

   is a linear isometry with respect to g.

   (c) Use this to show that the adjoint action

$$\begin{aligned} \mathrm{ad}_U &: \mathfrak{g} \to \mathfrak{g}, \\ \mathrm{ad}_U(X) &= [U,X] \end{aligned}$$

   is skew-symmetric, i.e.,

$$g([U,X],Y) = -g(X,[U,Y]).$$

   Hint: It is shown in the appendix that $U \to \mathrm{ad}_U$ is the differential of $h \to \mathrm{Ad}_h$. (See also chapter 3).

(6) Let V be an n-dimensional vector space with a symmetric nondegenerate bilinear form g of index p.
   (a) Show that there exists a basis $e_1, \ldots, e_n$ such that $g(e_i, e_j) = 0$ if $i \neq j$, $g(e_i, e_i) = -1$ if $i = 1, \ldots, p$ and $g(e_i, e_i) = 1$ if $i = p+1, \ldots, n$. Thus V is isometric to $\mathbb{R}^{p,q}$.

(b) Show that for any v we have the expansion

$$\begin{aligned} v &= \sum_{i=1}^{n} \frac{g(v,e_i)}{g(e_i,e_i)} e_i \\ &= -\sum_{i=1}^{p} g(v,e_i)\, e_i + \sum_{i=p+1}^{n} g(v,e_i)\, e_i. \end{aligned}$$

(c) Let $L : V \to V$ be a linear operator. Show that

$$\operatorname{tr}(L) = \sum_{i=1}^{n} \frac{g(L(e_i), e_i)}{g(e_i,e_i)}.$$

(b) Show that the *inner automorphism* $\mathrm{Ad}_h(x) = hxh^{-1}$ is a Riemannian isometry. Conclude that its differential at $x = e$ denoted by the same letters

$$\mathrm{Ad}_h : \mathfrak{g} \to \mathfrak{g}$$

is a linear isometry with respect to g.

Chapter 2 Curvature

1. Connections
2. The connection in Local Coordinates
3. Curvature
4. The Fundamental Curvature Equations
5. The Equations of Riemannian Geometry
6. Some Tensor Concepts
7. Further Study
8. Exercises

(1) Show that the connection on Euclidean space is the only affine connection such that $\nabla X = 0$ for all constant vector fields X.

(2) If $F : M \to M$ is a diffeomorphism, then the push-forward of a vector field is defined as

$$(F_* X)|_p = DF\left(X|_{F^{-1}(p)}\right).$$

Let F be an isometry on (M,g).

(a) Show that $F_*(\nabla_X Y) = \nabla_{F_* X} F_* Y$ for all vector fields.

(b) If $(M,g) = (\mathbb{R}^n, \mathrm{can})$, then isometries are of the form $F(x) = Ox + b$, where $O \in O(n)$ and $b \in \mathbb{R}^n$. Hint: Show that F maps constant vector fields to constant vector fields.

(3) Let G be a Lie group. Show that there is a unique affine connection such that $\nabla X = 0$ for all left invariant vector fields. Show that this connection is torsion free iff the Lie algebra is Abelian.

(4) Show that if X is a vector field of constant length on a Riemannian manifold, then $\nabla_v X$ is always perpendicular to X.

(5) For any $p \in (M, g)$ and orthonormal basis $e_1, \ldots, e_n$ for T_pM, show that there is an orthonormal frame $E_1, \ldots, E_n$ in a neighborhood of p such that $E_i = e_i$ and $(\nabla E_i)|_p = 0$. Hint: Fix an orthonormal frame $\bar{E}_i$ near $p \in M$ with $\bar{E}_i(p) = e_i$. If we define $E_i = \alpha_i^j \bar{E}_j$, where $\left[\alpha_i^j(x)\right] \in SO(n)$ and $\alpha_i^j(p) = \delta_i^j$, then this will yield the desired frame provided that the $D_{e_k}\alpha_i^j$ are appropriately prescribed.

(6) (Riemann) As in the previous problem, but now show that there are coordinates $x^1, \ldots, x^n$ such that $\partial_i = e_i$ and $\nabla\partial_i = 0$ at p. These conditions imply that the metric coefficients satisfy $g_{ij} = \delta_{ij}$ and $\partial_k g_{ij} = 0$ at p. Such coordinates are called normal coordinates at p. Show that in normal coordinates g viewed as a matrix function of x has the expansion

$$\begin{aligned} g &= \sum_{i,j=1}^{n} g_{ij}dx^i dx^j \\ &= \sum_{i=1}^{n} dx^i dx^i \\ &\quad + \sum_{i<j,k<l} R_{ijkl}\left(x^i dx^j - x^j dx^i\right)\left(x^k dx^l - x^l dx^k\right) + o\left(|x|^2\right), \end{aligned}$$

where $R_{ijkl} = g\left(R\left(\partial_i, \partial_j\right)\partial_k, \partial_l\right)(p)$. In dimension 2 this formula reduces to

$$\begin{aligned} g &= dx^2 + dy^2 + R_{1212}\left(xdy - ydx\right)^2 + o\left(x^2 + y^2\right) \\ &= dx^2 + dy^2 - \sec(p)\left(xdy - ydx\right)^2 + o\left(x^2 + y^2\right). \end{aligned}$$

(7) Let M be an n-dimensional submanifold of $\mathbb{R}^{n+m}$ with the induced metric and assume that we have a local coordinate system given by a parametrization $x^s\left(u^1, ..., u^n\right)$, $s = 1, ..., n+m$. Show that in these coordinates we have:

(a)

$$g_{ij} = \sum_{s=1}^{n+m} \frac{\partial x^s}{\partial u^i}\frac{\partial x^s}{\partial u^j}.$$

(b)

$$\Gamma_{ij,k} = \sum_{s=1}^{n+m} \frac{\partial x^s}{\partial u^k}\frac{\partial^2 x^s}{\partial u^i \partial u^j}.$$

(c) R_{ijkl} depends only on the first and second partials of x^s.

(8) Show that $\mathrm{Hess} f = \nabla df$.

(9) Let r be a distance function and $S(X) = \nabla_X \partial_r$ the $(1,1)$ version of the Hessian. Show that

$$\begin{aligned} L_{\partial_r} S &= \nabla_{\partial_r} S, \\ L_{\partial_r} S + S^2 &= -R_{\partial_r}. \end{aligned}$$

How do you reconcile this with what happens for the fundamental equations for the $(0,2)$-version of the Hessian?

(10) Let (M, g) be oriented and define the Riemannian volume form $d\text{vol}$ as follows:

$$d\text{vol}(v_1, \ldots, v_n) = \det(g(v_i, e_j)),$$

where $e_1, \ldots, e_n$ is a positively oriented orthonormal basis for T_pM.

(a) Show that if $v_1, \ldots, v_n$ is positively oriented, then

$$d\text{vol}(v_1, \ldots, v_n) = \sqrt{\det(g(v_i, v_j))}.$$

(b) Show that the volume form is parallel.

(c) Show that in positively oriented coordinates,

$$d\text{vol} = \sqrt{\det(g_{ij})} dx^1 \wedge \cdots \wedge dx^n.$$

(d) If X is a vector field, show that

$$L_X d\text{vol} = \text{div}(X) \, d\text{vol}.$$

(e) Conclude that the Laplacian has the formula

$$\Delta u = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l u \right).$$

Given that the coordinates are normal at p we get as in Euclidean space that

$$\Delta f(p) = \sum_{i=1}^{n} \partial_i \partial_i f.$$

(11) Let (M, g) be a oriented Riemannian manifold with volume form $d\text{vol}$ as above.

(a) If f has compact support, then

$$\int_M \Delta f \cdot d\text{vol} = 0.$$

(b) Show that

$$\text{div}(f \cdot X) = g(\nabla f, X) + f \cdot \text{div} X.$$

(c) Show that

$$\Delta(f_1 \cdot f_2) = (\Delta f_1) \cdot f_2 + 2g(\nabla f_1, \nabla f_2) + f_1 \cdot (\Delta f_2).$$

(d) Establish the integration by parts formula for functions with compact support:

$$\int_M f_1 \cdot \Delta f_2 \cdot d\mathrm{vol} = -\int_M g\left(\nabla f_1, \nabla f_2\right) \cdot d\mathrm{vol}.$$

(e) Conclude that if f is sub- or superharmonic (i.e., $\Delta f \geq 0$ or $\Delta f \leq 0$) then f is constant. (Hint: first show $\Delta f = 0$; then use integration by parts on $f \cdot \Delta f$.) This result is known as the *weak maximum principle*. More generally, one can show that any subharmonic (respectively superharmonic) function that has a global maximum (respectively minimum) must be constant. For this one does not need f to have compact support. This result is usually referred to as the *strong maximum principle*.

(12) A vector field and its corresponding flow is said to be *incompressible* if $\mathrm{div} X = 0$.
  (a) Show that X is incompressible iff the local flows it generates are volume preserving (i.e., leave the Riemannian volume form invariant).
  (b) Let X be a unit vector field X on $\mathbb{R}^2$. Show that $\nabla X = 0$ if X is incompressible.
  (c) Find a unit vector field X on $\mathbb{R}^3$ that is incompressible but where $\nabla X \neq 0$.

(13) Let X be a unit vector field on (M, g) such that $\nabla_X X = 0$.
  (a) Show that X is locally the gradient of a distance function iff the orthogonal distribution is integrable.
  (b) Show that X is the gradient of a distance function in a neighborhood of $p \in M$ iff the orthogonal distribution has an integral submanifold through p. Hint: It might help to show that $L_X \theta_X = 0$.
  (c) Find X with the given conditions so that it is not a gradient field. Hint: Consider S^3.

(14) Given an orthonormal frame $E_1, \ldots, E_n$ on (M, g), define the *structure constants* c_{ij}^k by $[E_i, E_j] = c_{ij}^k E_k$. Then define the Γs and Rs by

$$\begin{aligned} \nabla_{E_i} E_j &= \Gamma_{ij}^k E_k, \\ R\left(E_i, E_j\right) E_k &= R_{ijk}^l E_l \end{aligned}$$

and compute them in terms of the cs. Notice that on Lie groups with left-invariant metrics the structure constants can be assumed to be constant. In this case, computations simplify considerably.

(15) There is yet another effective method for computing the connection and curvatures, namely, the *Cartan formalism*. Let (M, g) be a Riemannian manifold. Given a frame $E_1, \dots, E_n$, the connection can be written

$$\nabla E_i = \omega_i^j E_j,$$

where ω_i^j are 1-forms. Thus,

$$\nabla_v E_i = \omega_i^j (v) E_j.$$

Suppose now that the frame is orthonormal and let ω^i be the dual coframe, i.e., $\omega^i (E_j) = \delta_j^i$. Show that the *connection forms* satisfy

$$\begin{aligned} \omega_i^j &= -\omega_j^i, \\ d\omega^i &= \omega^j \wedge \omega_j^i. \end{aligned}$$

These two equations can, conversely, be used to compute the connection forms given the orthonormal frame. Therefore, if the metric is given by declaring a certain frame to be orthonormal, then this method can be very effective in computing the connection.

If we think of $\left[\omega_i^j\right]$ as a matrix, then it represents a 1-form with values in the skew-symmetric $n \times n$ matrices, or in other words, with values in the Lie algebra $\mathfrak{so}(n)$ for $O(n)$.

The *curvature forms* Ω_i^j are 2-forms with values in $\mathfrak{so}(n)$. They are defined as

$$R(\cdot, \cdot) E_i = \Omega_i^j E_j.$$

Show that they satisfy

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \Omega_i^j.$$

When reducing to Riemannian metrics on surfaces we obtain for an orthonormal frame E_1, E_2 with coframe ω^1, ω^2

$$\begin{aligned} d\omega^1 &= \omega^2 \wedge \omega_2^1, \\ d\omega^2 &= -\omega^1 \wedge \omega_2^1, \\ d\omega_2^1 &= \Omega_2^1, \\ \Omega_2^1 &= \sec \cdot d\text{vol}. \end{aligned}$$

(16) Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. In chapter 3 it will be shown that the converse is not true, and also that a metric with parallel curvature tensor doesn't have to be Einstein.

(17) Show that if R is the $(1,3)$-curvature tensor and Ric the $(0,2)$-Ricci tensor, then

$$(\mathrm{div} R)(X,Y,Z) = (\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z).$$

Conclude that $\mathrm{div} R = 0$ if $\nabla \mathrm{Ric} = 0$. Then show that $\mathrm{div} R = 0$ iff the $(1,1)$ Ricci tensor satisfies:

$$(\nabla_X \mathrm{Ric})(Y) = (\nabla_Y \mathrm{Ric})(X) \text{ for all } X, Y.$$

(18) Let G be a Lie group with a bi-invariant metric. Using left-invariant fields establish the following formulas. Hint: First go back to the exercises to chapter 1 and take a peek at chapter 3 where some of these things are proved.

(a) $\nabla_X Y = \frac{1}{2}[X,Y]$.

(b) $R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$.

(c) $g(R(X,Y)Z,W) = -\frac{1}{4}(g([X,Y],[Z,W]))$. Conclude that the sectional curvatures are nonnegative.

(d) Show that the curvature operator is also nonnegative by showing that:

$$g\left(\mathfrak{R}\left(\sum_{i=1}^{k} X_i \wedge Y_i\right), \left(\sum_{i=1}^{k} X_i \wedge Y_i\right)\right) = \frac{1}{4}\left|\sum_{i=1}^{k} [X_i, Y_i]\right|^2.$$

(e) Show that $\mathrm{Ric}(X,X) = 0$ iff X commutes with all other left-invariant vector fields. Thus G has positive Ricci curvature if the center of G is discrete.

(f) Consider the linear map $\Lambda^2\mathfrak{g} \to [\mathfrak{g},\mathfrak{g}]$ that sends $X \wedge Y$ to $[X,Y]$. Show that the sectional curvature is positive iff this map is an isomorphism. Conclude that this can only happen if $n = 3$ and $\mathfrak{g} = \mathfrak{su}(2)$.

(19) It is illustrative to use the Cartan formalism in the above problem and compute all quantities in terms of the structure constants for the Lie algebra. Given that the metric is bi-invariant, it follows that with respect to an orthonormal basis they satisfy

$$c_{ij}^k = -c_{ji}^k = c_{jk}^i.$$

The first equality is skew-symmetry of the Lie bracket, and the second is bi-invariance of the metric.

(20) Suppose we have two Riemannian manifolds (M, g_M) and (N, g_N). Then the product has a natural product metric $(M \times N, g_M + g_N)$. Let X be a vector field on M and Y one on N, show that if we regard these as vector fields on $M \times N$, then $\nabla_X Y = 0$. Conclude that $\sec(X,Y) = 0$. This means that product metrics always have many curvatures that are zero.

(21) Suppose we have two distributions E and F on (M,g), that are orthogonal complements of each other in TM. In addition, assume that the distributions are parallel i.e., if two vector fields X and Y are tangent to, say, E, then $\nabla_X Y$ is also tangent to E.
(a) Show that the distributions are integrable.
(b) Show that around any point in M there is a product neighborhood $U = V_E \times V_F$ such that $(U,g) = (V_E \times V_F, g|_E + g|_F)$, where $g|_E$ and $g|_F$ are the restrictions of g to the two distributions. In other words, M is locally a product metric.

(22) Let X be a parallel vector field on (M,g). Show that X has constant length. Show that X generates parallel distributions, one that contains X and the other that is the orthogonal complement to X. Conclude that locally the metric is a product with an interval $(U,g) = \left(V \times I, g|_{TV} + dt^2\right)$.

(23) For 3-dimensional manifolds, show that if the curvature operator in diagonal form looks like

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

then the Ricci curvature has a diagonal form like

$$\begin{pmatrix} \alpha+\beta & 0 & 0 \\ 0 & \beta+\gamma & 0 \\ 0 & 0 & \alpha+\gamma \end{pmatrix}.$$

Moreover, the numbers α, β, γ must be sectional curvatures.

(24) The *Einstein tensor* on a Riemannian manifold is defined as

$$G = \mathrm{Ric} - \frac{\mathrm{scal}}{2} \cdot I.$$

Show that $G = 0$ in dimension 2 and that $\mathrm{div}G = 0$ in higher dimensions. This tensor is supposed to measure the mass/energy distribution. The fact that it is divergence free tells us that energy and momentum are conserved. In a vacuum, one therefore imagines that $G = 0$. Show that this happens in dimensions > 2 iff the metric is Ricci flat.

(25) This exercise will give you a way of finding the curvature tensor from the sectional curvatures. Using the Bianchi identity show that

$$\begin{aligned} -6R(X,Y,Z,W) &= \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=t=0} \{R(X+sZ, Y+tW, Y+tW, X+sZ) \\ &\quad -R(X+sW, Y+tZ, Y+tZ, X+sW)\}. \end{aligned}$$

(26) Using polarization show that the norm of the curvature operator on $\Lambda^2 T_p M$ is bounded by

$$|\mathfrak{R}|_p| \leq c(n)\, |\sec|_p$$

for some constant $c(n)$ depending on dimension, and where $|\sec|_p$ denotes the largest absolute value for any sectional curvature of a plane in $T_p M$.

(31) Recall that complex manifolds have complex tangent spaces. Thus we can multiply vectors by $\sqrt{-1}$. As a generalization of this we can define an *almost complex* structure. This is a $(1,1)$-tensor J such that $J^2 = -I$. Show that the *Nijenhuis tensor:*

$$N(X,Y) = [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X,Y]$$

is indeed a tensor. If J comes from a complex structure then $N = 0$, conversely Newlander&Nirenberg have shown that J comes from a complex structure if $N = 0$.

A *Hermitian structure* on a Riemannian manifold (M, g) is an almost complex structure J such that

$$g(J(X), J(Y)) = g(X,Y).$$

The *Kähler form* of a Hermitian structure is

$$\omega(X,Y) = g(J(X), Y).$$

Show that ω is a 2-form. Show that $d\omega = 0$ iff $\nabla J = 0$. If the Kähler form is closed, then we call the metric a Kähler metric.

Chapter 3 Examples

(1) Show that the Schwarzschild metric doesn't have parallel curvature tensor.
(2) Show that the Berger spheres ($\varepsilon \neq 1$) do not have parallel curvature tensor.
(3) Show that $\mathbb{C}P^2$ has parallel curvature tensor.

(4) The *Heisenberg group* with its Lie algebra is

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\},$$
$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A basis for the Lie algebra is:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) Show that the only nonzero brackets are

$$[X, Y] = -[Y, X] = Z.$$

Now introduce a left-invariant metric on G such that X, Y, Z form an orthonormal frame.

(b) Show that the Ricci tensor has both negative and positive eigenvalues.
(c) Show that the scalar curvature is constant.
(d) Show that the Ricci tensor is not parallel.

(5) Let $\tilde{g} = e^{2\psi} g$ be a metric conformally equivalent to g. Show that
(a)

$$\tilde{\nabla}_X Y = \nabla_X Y + ((D_X \psi) Y + (D_Y \psi) X - g(X, Y) \nabla \psi)$$

(b) If X, Y are orthonormal with respect to g, then

$$e^{2\psi} \widetilde{\sec}(X, Y) = \sec(X, Y) - \text{Hess}\psi(X, X) - \text{Hess}\psi(Y, Y) - |\nabla\psi|^2 + (D_X\psi)^2 + (D_Y\psi)^2$$

(6) (a) Show that there is a family of Ricci flat metrics on TS^2 of the form

$$dr^2 + \varphi^2(r)\left(\psi^2(r)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right),$$

$$\dot{\varphi} = \psi,$$
$$\dot{\varphi}^2 = 1 - k\varphi^{-4},$$
$$\varphi(0) = k^{\frac{1}{4}}, \dot{\varphi}(0) = 0,$$
$$\psi(0) = 0, \dot{\psi}(0) = 2.$$

(b) Show that $\varphi(r) \sim r$, $\dot{\varphi}(r) \sim 1$, $\ddot{\varphi}(r) \sim 2kr^{-5}$ as $r \to \infty$. Conclude that all curvatures are of order r^{-6} as $r \to \infty$ and that the metric looks like $(0, \infty) \times \mathbb{R}P^3 = (0, \infty) \times SO(3)$ at infinity. Moreover, show that scaling one of these metrics corresponds to changing k. Thus, we really have only one Ricci flat metric; it is called the *Eguchi-Hanson metric*.

(7) For the general metric

$$dr^2 + \varphi^2(r)\left(\psi^2(r)(\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2\right)$$

show that the $(1,1)$-tensor, which in the orthonormal frame looks like

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

yields a Hermitian structure.

(a) Show that this structure is Kähler, i.e., parallel, iff $\dot{\varphi} = \psi$.
(b) Find the scalar curvature for such metrics.
(c) Show that there are scalar flat metrics on all the 2-dimensional vector bundles over S^2. The one on TS^2 is the Eguchi-Hanson metric, and the one on $S^2 \times \mathbb{R}^2$ is the Schwarzschild metric.

(8) Show that $\tau\left(\mathbb{R}P^{n-1}\right)$ admits rotationally symmetric metrics

$$dr^2 + \varphi^2(r)\, ds^2_{n-1}$$

such that $\varphi(r) = r$ for $r > 1$ and the Ricci curvatures are nonpositive. Thus, the Euclidean metric can be topologically perturbed to have nonpositive Ricci curvature. It is not possible to perturb the Euclidean metric in this way to have nonnegative scalar curvature or nonpositive sectional curvature. Try to convince yourself of that by looking at rotationally symmetric metrics on $\mathbb{R}^n$ and $\tau\left(\mathbb{R}P^{n-1}\right)$.

(9) A Riemannian manifold (M, g) is said to be *locally conformally flat* if every $p \in M$ lies in a coordinate neighborhood U such that

$$g = \varphi^2\left(\left(dx^1\right)^2 + \cdots + (dx^n)^2\right).$$

(a) Show that the space forms S^n_k are locally conformally flat.
(b) With some help from the literature, show that any 2-dimensional Riemannian manifold is locally conformally flat (isothermal coordinates). In fact, any metric on a closed surface is conformal to a metric of constant curvature. This is called the *uniformization theorem*.
(c) Show that if an Einstein metric is locally conformally flat, then it has constant curvature.

(10) We say that (M, g) admits *orthogonal coordinates* around $p \in M$ if we have coordinates on some neighborhood of p, where

$$g_{ij} = 0 \text{ for } i \neq j,$$

i.e., the coordinate vector fields are perpendicular. Show that such coordinates always exist in dimension 2, while they may not exist in dimension > 3. To find a counterexample, you may want to show that in such coordinates the curvatures $R^l_{ijk} = 0$ if all indices are distinct. What about 3 dimensions?

Chapter 4 Hypersurfaces

1. The Gauss Map
2. Existence of Hypersurfaces
3. The Gauss-Bonnet Theorem
4. Further Study
5. Exercises

(1) Consider the hypersurface given by the graph $x^{n+1} = f(x^n)$, where $f : \mathbb{R} \to \mathbb{R}$ is smooth. Show that the shape operator doesn't necessarily vanish but that the hypersurface is isometric to $\mathbb{R}^n$.

(2) If X is a Killing field on an abstract surface (M^2, g) show that the index of any isolated zero is 1.

(3) Assume that we have a Riemannian immersion of an n-manifold into $\mathbb{R}^{n+1}$. If $n \geq 3$, then show that it can't have negative curvature. If $n = 2$ give an example where it does have negative curvature.

(4) Let (M, g) be a closed Riemannian n-manifold, and suppose that there is a Riemannian embedding into $\mathbb{R}^{n+1}$. Show that there must be a point $p \in M$ where the curvature operator $\mathfrak{R} : \Lambda^2 T_p M \to \Lambda^2 T_p M$ is positive. (Hint: Consider $f(x) = |x|^2$ and restrict it to M, then check what happens at a maximum.)

(5) Suppose (M, g) is immersed as a hypersurface in $\mathbb{R}^{n+1}$, with shape operator S.

(a) Using the Codazzi-Mainardi equations, show that

$$\text{div} S = d\,(\text{tr} S)\,.$$

(b) Show that if $S = f(x) \cdot I$ for some function f, then f must be a constant and the hypersurface must have constant curvature.

(c) Show that $S = \lambda \cdot \text{Ric}$ iff the metric has constant curvature.

(6) Let g be a metric on S^2 with curvature ≤ 1. Use the Gauss-Bonnet formula to show that $\text{vol}(S^2, g) \geq \text{vol} S^2(1) = 4\pi$.

Show that such a result cannot hold on S^3 by considering the Berger metrics.

(7) Assume that we have an orientable Riemannian manifold with nonzero Euler characteristic and $|\mathfrak{R}| \leq 1$. Find a lower bound for $\text{vol}(M, g)$. The one sided curvature bound that we used on surfaces does not suffice in higher dimensions, as one-sided curvature bounds do not necessarily imply one sided bounds on the Chern-Gauss-Bonnet integrand.

(8) Show that in even dimensions, orientable manifolds with positive (or nonnegative) curvature operator have positive (nonnegative) Euler characteristic. Conclude that if in addition, such manifolds have bounded curvature operator, then they have volume bounded from below. What happens when the curvature operator is nonpositive or negative?

(9) In dimension 4 show, using the exercises from chapter 3, that

$$\frac{1}{8\pi^2}\int_M \left(|R|^2 - \left|\text{Ric} - \frac{\text{scal}}{4}g\right|^2\right) = \frac{1}{8\pi^2}\int_M \text{tr}\left(A^2 - 2BB^* + C^2\right).$$

It was shown by Allendoerfer and Weil that in dimension 4

$$\chi(M) = \frac{1}{8\pi^2}\int_M \left(|R|^2 - \left|\text{Ric} - \frac{\text{scal}}{4}g\right|^2\right).$$

You can try to prove this using the above definition of K. If the metric is Einstein, show that

$$\begin{aligned}\chi(M) &= \frac{1}{8\pi^2}\int_M \text{tr}\left(A^2 - 2BB^* + C^2\right)\\ &= \frac{1}{8\pi^2}\int_M \left(\left|W^+\right|^2 + \left|W^-\right|^2 + \frac{\text{scal}^2}{24}\right).\end{aligned}$$

Chapter 5 Geodesics and Distance

1. Mixed Partials
2. Geodesics
3. The metric Structure of Riemannian Manifold
4. First Variation of Energy
5. The Exponential Map
6. Why Short Geodesics Are Segments
7. Local Geometry in Constant Curvature
8. Completeness
9. Characterization of Segments
10. Riemannian Isometries
11. Further Study
12. Exercises

(1) Assume that (M, g) has the property that all geodesics exist for a fixed time $\varepsilon > 0$. Show that (M, g) is geodesically complete.

(2) A Riemannian manifold is said to be homogeneous if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

(3) Assume that we have coordinates in a Riemannian manifold so that $g_{1i} = \delta_{1i}$. Show that x^1 is a distance function.

(4) Let γ be a geodesic in a Riemannian manifold (M, g). Let g' be another Riemannian metric on M with the properties: $g'(\dot\gamma, \dot\gamma) = g(\dot\gamma, \dot\gamma)$ and $g'(X, \dot\gamma) = 0$ iff $g(X, \dot\gamma) = 0$. Show that γ is also a geodesic with respect to g'.

(5) Show that if we have a vector field X on a Riemannian manifold (M, g) that vanishes at $p \in M$, then for any tensor T we have $L_X T = \nabla_X T$ at p. Conclude that the Hessian of a function is independent of the metric at a critical point. Can you find an interpretation of $L_X T$ at p?

(6) Show that any Riemannian manifold (M, g) admits a conformal change $(M, \lambda^2 g)$, where $\lambda : M \to (0, \infty)$, such that $(M, \lambda^2 g)$ is complete.

(7) On an open subset $U \subset \mathbb{R}^n$ we have the induced distance from the Riemannian metric, and also the induced distance from $\mathbb{R}^n$. Show that the two can agree even if U isn't convex.

(8) Let $N \subset (M, g)$ be a submanifold. Let ∇^N denote the connection on N that comes from the metric induced by g. Define the second fundamental form of N in M by

$$\mathrm{II}(X, Y) = \nabla^N_X Y - \nabla_X Y$$

(a) Show that $\mathrm{II}(X, Y)$ is symmetric and hence tensorial in X and Y.
(b) Show that $\mathrm{II}(X, Y)$ is always normal to N.
(c) Show that $\mathrm{II} = 0$ on N iff N is totally geodesic.
(d) If R^N is the curvature tensor for N, then

$$\begin{aligned} g(R(X,Y)Z, W) &= g\left(R^N(X,Y)Z, W\right) \\ &\quad - g(\mathrm{II}(Y,Z), \mathrm{II}(X,W)) + g(\mathrm{II}(X,Z), \mathrm{II}(Y,W)). \end{aligned}$$

(9) Let $f : (M, g) \to \mathbb{R}$ be a smooth function on a Riemannian manifold.
(a) If $\gamma : (a, b) \to M$ is a geodesic, compute the first and second derivatives of $f \circ \gamma$.
(b) Use this to show that at a local maximum (or minimum) for f the gradient is zero and the Hessian nonpositive (or nonnegative).
(c) Show that f has everywhere nonnegative Hessian iff $f \circ \gamma$ is convex for all geodesics γ in (M, g).

(10) Let $N \subset M$ be a submanifold of a Riemannian manifold (M, g).
(a) The distance from N to $x \in M$ is defined as

$$d(x, N) = \inf \{ d(x, p) : p \in N \}.$$

A unit speed curve $\sigma : [a, b] \to M$ with $\sigma(a) \in N, \sigma(b) = x$, and $\ell(\sigma) = d(x, N)$ is called a segment from x to N. Show that σ is also a segment from N to any $\sigma(t)$, $t < b$. Show that $\dot\sigma(a)$ is perpendicular to N.

(b) Show that if N is a closed subspace of M and (M, g) is complete, then any point in M can be joined to N by a segment.

(c) Show that in general there is an open neighborhood of N in M where all points are joined to N by segments.

(d) Show that $d(\cdot, N)$ is smooth on a neighborhood of N and that the integral curves for its gradient are the geodesics that are perpendicular to N.

(11) Compute the cut locus on a square torus $\mathbb{R}^2/\mathbb{Z}^2$.

(12) Compute the cut locus on a sphere and real projective space with the constant curvature metrics.

(13) In a metric space (X, d) one can measure the length of continuous curves $\gamma : [a, b] \to X$ by

$$\ell(\gamma) = \sup\left\{\sum d(\gamma(t_i), \gamma(t_{i+1})) : a = t_1 \le t_2 \le \cdots \le t_{k-1} \le t_k = b\right\}.$$

(a) Show that a curve has finite length iff it is absolutely continuous. Hint: Use the characterization that $\gamma : [a, b] \to X$ is absolutely continuous if and only if for each $\varepsilon > 0$ there is a $\delta > 0$ so that $\sum d(\gamma(s_i), \gamma(s_{i+1})) \le \varepsilon$ provided $\sum |s_i - s_{i+1}| \le \delta$.

(b) Show that this definition gives back our previous definition for smooth curves on Riemannian manifolds.

(c) Let $\gamma : [a, b] \to M$ be an absolutely continuous curve whose length is $d(\gamma(a), \gamma(b))$. Show that $\gamma = \sigma \circ \varphi$ for some segment σ and reparametrization φ.

(14) Show that in a Riemannian manifold,

$$d\left(\exp_p(tv), \exp_p(tw)\right) = |t| \cdot |v - w| + O\left(t^2\right).$$

(15) Assume that we have coordinates x^i around a point $p \in (M, g)$ such that $x^i(p) = 0$ and $g_{ij}x^j = x^i$. Show that these must be exponential coordinates. Hint: Define

$$r = \sqrt{(x^1)^2 + \cdots + (x^n)^2}$$

and show that it is a smooth distance function away from p, and that the integral curves for the gradient are geodesics emanating from p.

(16) If $N_1, N_2 \subset M$ are totally geodesic submanifolds, show that each component of $N_1 \cap N_2$ is a submanifold which is totally geodesic. Hint: The potential tangent space at $p \in N_1 \cap N_2$ should be $T_pN_1 \cap T_pN_2$.

(17) Show that for a complete manifold the functional distance is the same as the distance. What about incomplete manifolds?

(18) Let $\gamma : [0,1] \to M$ be a geodesic such that $\exp_{\gamma(0)}$ is regular at all $t\dot{\gamma}(0)$, for $t \leq 1$. Show that γ is a local minimum for the energy functional. Hint: Show that the lift of γ via $\exp_{\gamma(0)}$ is a minimizing geodesic in a suitable metric.

(19) Show, using the exercises on Lie groups from chapters 1 and 2, that on a Lie group G with a bi-invariant metric the geodesics through the identity are exactly the homomorphisms $\mathbb{R} \to G$. Conclude that the Lie group exponential map coincides with the exponential map generated by the bi-invariant Riemannian metric. Hint: First show that homomorphisms $\mathbb{R} \to G$ are precisely the integral curves for left invariant vector fields through $e \in G$.

(20) Repeat the previous exercise assuming that the metric is a bi-invariant semi-Riemannian metric. Show that the matrix group $Gl_n(\mathbb{R})$ of invertible $n \times n$ matrices admits a bi-invariant semi-Riemannian metric. Hint: for $X, Y \in T_I Gl_n(\mathbb{R})$ define

$$g(X,Y) = -\mathrm{tr}(XY).$$

(21) Construct a Riemannian metric on the tangent bundle to a Riemannian manifold (M,g) such that $\pi : TM \to M$ is a Riemannian submersion and the metric restricted to the tangent spaces is the given Euclidean metric.

(22) For a Riemannian manifold (M,g) let FM be the frame bundle of M. This is a fiber bundle $\pi : FM \to M$ whose fiber over $p \in M$ consists of orthonormal bases for T_pM. Find a Riemannian metric on FM that makes π into a Riemannian submersion and such that the fibers are isometric to $O(n)$.

(23) Show that a Riemannian submersion is a submetry.

(24) (Hermann) Let $f : (M,\bar{g}) \to (N,g)$ be a Riemannian submersion.
  (a) Show that (N,g) is complete if $(M,\bar{g})$ is complete.
  (b) Show that f is a fibration if $(M,\bar{g})$ is complete i.e., for every $p \in N$ there is a neighborhood $p \in U$ such that $f^{-1}(U)$ is diffeomorphic to $U \times f^{-1}(p)$. Give a counterexample when $(M,\bar{g})$ is not complete.

Chapter 6 Sectional Curvature Comparison I

(1) Show that in even dimensions the sphere and real projective space are the only closed manifolds with constant positive curvature.

(2) Suppose we have a rotationally symmetric metric $dr^2 + \varphi^2(r)\,d\theta^2$. We wish to understand parallel translation along a latitude, i.e., a curve with $r = a$. To do this we construct a cone $dr^2 + (\varphi(a) + \dot{\varphi}(a)(r-a))^2\,d\theta^2$ that is tangent to this surface at the latitude $r = a$. In case the surface really is a surface of revolution, this cone is a real cone that is tangent to the surface along the latitude $r = a$.

(a) Show that in the standard coordinates (r, θ) on these surfaces, the covariant derivative ∇_{∂_θ} is the same along the curve $r = a$. Conclude that parallel translation is the same along this curve on these two surfaces.

(b) Now take a piece of paper and try to figure out what parallel translations along a latitude on a cone looks like. If you unwrap the paper it is flat; thus parallel translation is what it is in the plane. Now rewrap the paper and observe that parallel translation along a latitude does not necessarily generate a closed parallel field.

(c) Show that in the above example the parallel field along $r = a$ closes up if $\dot{\varphi}(a) = 0$.

(3) *(Fermi-Walker transport)* Related to parallel transport there is a more obscure type of transport that is sometimes used in physics. Let $\gamma : [a, b] \to M$ be a curve into a Riemannian manifold whose speed never vanishes and

$$T = \frac{\dot{\gamma}}{|\dot{\gamma}|}$$

the unit tangent of γ. We say that V is a Fermi-Walker field along γ if

$$\begin{aligned} \dot{V} &= g(V, T)\,\dot{T} - g\left(V, \dot{T}\right) T \\ &= \left(\dot{T} \wedge T\right)(V). \end{aligned}$$

(a) Show that given $V(t_0)$ there is a unique Fermi-Walker field V along γ whose value at t_0 is $V(t_0)$.

(b) Show that T is a Fermi-Walker field along γ.

(c) Show that if V, W are Fermi-Walker fields along γ, then $g(V, W)$ is constant along γ.

(d) If γ is a geodesic, then Fermi-Walker fields are parallel.

(4) Let (M, g) be a complete n-manifold of constant curvature k. Select a linear isometry $L : T_pM \to T_{\bar{p}}S_k^n$. When $k \leq 0$ show that

$$\exp_p \circ L^{-1} \circ \exp_{\bar{p}}^{-1} : S_k^n \to M$$

is a Riemannian covering map. When $k > 0$ show that

$$\exp_p \circ L^{-1} \circ \exp_{\bar{p}}^{-1} : S_k^n - \{-\bar{p}\} \to M$$

extends to a Riemannian covering map $S_k^n \to M$. (Hint: Use that the differential of the exponential maps is controlled by the metric, which in turn can be computed when the curvature is constant. You should also use the conjugate radius ideas presented in connection with the Hadamard-Cartan theorem.)

(5) Let $\gamma : [0,1] \to M$ be a geodesic. Show that $\exp_{\gamma(0)}$ has a critical point at $t\dot{\gamma}(0)$ iff there is a Jacobi field J along γ such that $J(0) = 0$, $\dot{J}(0) \neq 0$, and $J(t) = 0$.

(6) Let $\gamma(s,t) : [0,1]^2 \to (M,g)$ be a variation such that $R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right) = 0$ everywhere. Show that for each $v \in T_{\gamma(0,0)}M$, there is a parallel field $V : [0,1]^2 \to TM$ along γ, i.e., $\frac{\partial V}{\partial s} = \frac{\partial V}{\partial t} = 0$ everywhere.

(7) Using

$$R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right)\frac{\partial \gamma}{\partial u} = \frac{\partial^3 \gamma}{\partial s \partial t \partial u} - \frac{\partial^3 \gamma}{\partial t \partial s \partial u}$$

show that the two skew-symmetry properties and Bianchi's first identity hold for the curvature tensor.

(8) Let γ be a geodesic and X a Killing field in a Riemannian manifold. Show that the restriction of X to γ is a Jacobi field.

(9) Let γ be a geodesic in a Riemannian manifold and J_1, J_2 Jacobi fields along γ. Show that

$$g\left(\dot{J}_1, J_2\right) - g\left(J_1, \dot{J}_2\right)$$

is constant. A special case is when $J_2 = \dot{\gamma}$.

(10) A Riemannian manifold is said to be *k-point homogeneous* if for all pairs of points $(p_1, \ldots, p_k)$ and $(q_1, \ldots, q_k)$ with $d(p_i, p_j) = d(q_i, q_j)$ there is an isometry F with $F(p_i) = q_i$. When $k = 1$ we simply say that the space is homogeneous.

(a) Show that a homogenous space has constant scalar curvature.

(b) Show that if $k > 2$ and (M,g) is k-point homogeneous, then M is also $(k-1)$-point homogeneous.

(c) Show that if (M,g) is two-point homogeneous, then (M,g) is an Einstein metric.

(d) Show that if (M,g) is three-point homogeneous, then (M,g) has constant curvature.

(e) Classify all three-point homogeneous spaces. Hint: The only one that isn't simply connected is the real projective space.

(11) Show that if G is an infinite Abelian group that is the subgroup of the fundamental group of a manifold with constant curvature, then either the manifold is flat or G is cyclic.

(12) Let $M \to N$ be a Riemannian k-fold covering map. Show, $\mathrm{vol}M = k \cdot \mathrm{vol}N$.

(13) Starting with a geodesic on a two-dimensional space form, discuss how the equidistant curves change as they move away from the original geodesic.

(14) Introduce polar coordinates $(r, \theta) \in (0, \infty) \times S^{n-1}$ on a neighborhood U around a point $p \in (M, g)$. If (M, g) has $\sec \geq 0$ ($\sec \leq 0$), show that any curve $\gamma(t) = (r(t), \theta(t))$ is shorter (longer) in the metric g than in the Euclidean metric on U.

(15) Around an orientable hypersurface $H \hookrightarrow (M, g)$ introduce the usual coordinates $(r, x) \in \mathbb{R} \times H$ on some neighborhood U around H. On U we have aside from the given metric g, also the radially flat metric $dt^2 + g_0$, where g_0 is the restriction of g to H. If M has $\sec \geq 0$ ($\sec \leq 0$) and $\gamma(t) = (r(t), x(t))$ is a curve, where $r \geq 0$ and the shape operator is ≤ 0 (≥ 0) at $x(t)$ for all t, show that γ is shorter (longer) with respect to g than with respect to the radially flat metric $dt^2 + g_0$.

(16) *(Frankel)* Let M be an n-dimensional Riemannian manifold of positive curvature and A, B two totally geodesic submanifolds. Show that A and B must intersect if $\mathrm{dim}A + \mathrm{dim}B \geq n - 1$. Hint: assume that A and B do not intersect. Then find a segment of shortest length from A to B. Show that this segment is perpendicular to each submanifold. Then use the dimension condition to find a parallel field along this geodesic that is tangent to A and B at the endpoints to the segments. Finally use the second variation formula to get a shorter curve from A to B.

(17) Let M be a complete n-dimensional Riemannian manifold and $A \subset M$ a compact submanifold. Without using Wilking's connectedness principle establish the following statements directly.
   (a) Show that curves in $\Omega_{A,A}(M)$ that are not stationary for the energy functional can be deformed to shorter curves in $\Omega_{A,A}(M)$.
   (b) Show that the stationary curves for the energy functional on $\Omega_{A,A}(M)$ consists of geodesics that are perpendicular to A at the end points.
   (c) If M has positive curvature, $A \subset M$ is totally geodesic, and $2\mathrm{dim}A \geq \mathrm{dim}M$, then the stationary curves can be deformed to shorter curves in $\Omega_{A,A}(M)$.
   (d) *(Wilking)* Conclude from c. that any curve $\gamma : [0, 1] \to M$ that starts and ends in A is homotopic through such curves to a curve in A, i.e., $\pi_1(M, A)$ is trivial.

(18) Generalize Preissmann's theorem to show that any solvable subgroup of the fundamental group must be cyclic.

(19) Let (M,g) be an oriented manifold of positive curvature and suppose we have an isometry $F : M \to M$ of finite order without fixed points. Show that if $\dim M$ is even, then F must be orientation reversing, while if $\dim M$ is odd, it must be orientation preserving. Weinstein has proven that this holds even if we don't assume that F has finite order.

(20) Use an analog of Cartan's result on isometries of finite order in nonpositive curvature to show that any closed manifold of constant curvature $= 1$ must either be the standard sphere or have diameter $\leq \frac{\pi}{2}$. Generalize this to show that any closed manifold with $\sec \geq 1$ is either simply connected or has diameter $\leq \frac{\pi}{2}$. In chapter 11 we shall show the stronger statement that a closed manifold with $\sec \geq 1$ and diameter $> \frac{\pi}{2}$ must in fact be homeomorphic to a sphere.

(21) *(The Index Form)* Below we shall use the second variation formula to prove several results established in chapter 5. If V, W are vector fields along a geodesic $\gamma : [0,1] \to (M,g)$, then the *index form* is the symmetric

bilinear form

$$I_0^1(V,W) = I(V,W) = \int_0^1 \left(g\left(\dot{V}, \dot{W}\right) - g\left(R\left(V,\dot{\gamma}\right)\dot{\gamma}, W\right)\right) dt.$$

In case the vector fields come from a proper variation of γ this is equal to the second variation of energy. Assume below that $\gamma : [0,1] \to (M,g)$ locally minimizes the energy functional. This implies that $I(V,V) \geq 0$ for all proper variations.

(a) If $I(V,V) = 0$ for a proper variation, then V is a Jacobi field. Hint: Let W be any other variational field that also vanishes at the end points and use that

$$0 \leq I(V+\varepsilon W, V+\varepsilon W) = I(V,V) + 2\varepsilon I(V,W) + \varepsilon^2 I(W,W)$$

for all small ε to show that $I(V,W) = 0$. Then use that this holds for all W to show that V is a Jacobi field.

(b) Let V and J be variational fields along γ such that $V(0) = J(0)$ and $V(1) = J(1)$. If J is a Jacobi field show that

$$I(V,J) = I(J,J).$$

(c) *(The Index Lemma)* Assume in addition that there are no Jacobi fields along γ that vanish at both end points. If V and J are as in b. show that $I(V,V) \geq I(J,J)$ with equality holding only if $V = J$ on $[0,1]$. Hint: Prove that if $V \neq J$, then

$$0 < I(V-J, V-J) = I(V,V) - I(J,J).$$

(d) Assume that there is a nontrivial Jacobi field J that vanishes at 0 and 1, show that $\gamma : [0, 1+\varepsilon] \to M$ is not locally minimizing for $\varepsilon > 0$. Hint: For sufficiently small ε there is a Jacobi field $K : [1-\varepsilon, 1+\varepsilon] \to TM$ such that $K(1+\varepsilon) = 0$ and $K(1-\varepsilon) = J(1-\varepsilon)$. Let V be the variational field such that $V|_{[0,1-\varepsilon]} = J$ and $V|_{[1-\varepsilon,1+\varepsilon]} = K$. Finally extend J to be zero on $[1, 1+\varepsilon]$. Now show that

$$\begin{aligned} 0 &= I_0^1(J,J) = I_0^{1+\varepsilon}(J,J) = I_0^{1-\varepsilon}(J,J) + I_{1-\varepsilon}^{1+\varepsilon}(J,J) \\ &> I_0^{1-\varepsilon}(J,J) + I_{1-\varepsilon}^{1+\varepsilon}(K,K) = I(V,V). \end{aligned}$$

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