An introduction to Riemannian Geometry Jose Natari

Ch 5 Geometric Mechanics

5.1 Mechanical Systems

Exercise 1.16

(1) Generalize Examples 1.1, 1.4, and 1.8 to a system of k particles moving in $\mathbb{R}^n$.

(2) Let $(M, \langle\cdot,\cdot\rangle, \mathcal{F})$ be a mechanical system. Show that the Newton equation defines a flow on TM, generated by the vector field $X \in \mathfrak{X}(TM)$ whose local expression is

$$X = v^i \frac{\partial}{\partial x^i} + \left(\sum_{j=1}^{n} g^{ij}(x) F_j(x, v) - \sum_{j,k=1}^{n} \Gamma^i_{jk}(x) v^j v^k \right) \frac{\partial}{\partial v^i},$$

where $(x^1, \ldots, x^n)$ are local coordinates on M, $(x^1, \ldots, x^n, v^1, \ldots, v^n)$ are the local coordinates induced on TM, and

$$\mathcal{F} = \sum_{i=1}^{n} F_i(x, v) dx^i$$

on these coordinates. What are the fixed points of this flow?

(3) (*Harmonic oscillator*) The **harmonic oscillator** (in appropriate units) is the conservative mechanical system $(\mathbb{R}, dx \otimes dx, -dU)$, where $U : \mathbb{R} \to \mathbb{R}$ is given by

$$U(x) := \frac{1}{2}\omega^2 x^2.$$

(a) Write the equation of motion and its general solution.

(b) Friction can be included in this model by considering the external force

$$\mathcal{F}\left(u \frac{d}{dx}\right) = -dU - 2ku\, dx$$

(where $k > 0$ is a constant). Write the equation of motion of this new mechanical system and its general solution.

(c) Generalize (a) to the n-dimensional harmonic oscillator, whose potential energy $U : \mathbb{R}^n \to \mathbb{R}$ is given by

$$U(x^1, \dots, x^n) := \frac{1}{2}\omega^2 \left(\left(x^1\right)^2 + \cdots + \left(x^n\right)^2 \right).$$

(4) Consider the conservative mechanical system $(\mathbb{R}, dx \otimes dx, -dU)$. Show that:

(a) the flow determined by the Newton equation on $T\mathbb{R} \cong \mathbb{R}^2$ is generated by the vector field

$$X = v\frac{\partial}{\partial x} - U'(x)\frac{\partial}{\partial v} \in \mathfrak{X}(\mathbb{R}^2);$$

(b) the fixed points of the flow are the points of the form $(x_0, 0)$, where x_0 is a critical point of U;

(c) if x_0 is a maximum of U with $U''(x_0) < 0$ then $(x_0, 0)$ is an unstable fixed point;

(d) if x_0 is a minimum of U with $U''(x_0) > 0$ then $(x_0, 0)$ is a stable fixed point, with arbitrarily small neighborhoods formed by periodic orbits.

(e) the periods of these orbits converge to $2\pi U''(x_0)^{-\frac{1}{2}}$ as they approach $(x_0, 0)$;

(f) locally, any conservative mechanical system $(M, \langle \cdot, \cdot \rangle, -dU)$ with $\dim M = 1$ is of the form above.

(5) Prove Lemma 1.12. (**Hint:** Use the Koszul formula).

(6) Prove Lemma 1.13.

(7) If $(M, \langle \cdot, \cdot \rangle)$ is a compact Riemannian manifold, it is known that there exists a nontrivial periodic geodesic. Use this fact to show that if M is compact then any conservative mechanical system $(M, \langle \cdot, \cdot \rangle, -dU)$ admits a nontrivial periodic motion.

(8) Prove Proposition 1.14.

(9) Recall that the hyperbolic plane is the upper half plane

$$H = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}$$

with the Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{1}{y^2}\,(dx \otimes dx + dy \otimes dy)$$

[cf. Exercise 3.3(5) in Chap. 3]. Use Proposition 1.14 to compute the Christoffel symbols for the Levi–Civita connection of $(H, \langle \cdot, \cdot \rangle)$ in the coordinates (x, y).

(10) (*Kepler problem*) The **Kepler problem** (in appropriate units) consists in determining the motion of a particle of mass $m = 1$ in the central potential

$$U = -\frac{1}{r}.$$

(a) Show that the equations of motion can be integrated to

$$r^2\dot{\theta} = p_\theta,$$
$$\frac{\dot{r}^2}{2} + \frac{{p_\theta}^2}{2r^2} - \frac{1}{r} = E,$$

where E and p_θ are integration constants.

(b) Use these equations to show that $u = \frac{1}{r}$ satisfies the linear ODE

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{{p_\theta}^2}.$$

(c) Assuming that the **pericenter** (i.e. the point in the particle's orbit closer to the center of attraction $r = 0$) occurs at $\theta = 0$, show that the equation of the particle's trajectory is

$$r = \frac{{p_\theta}^2}{1 + \varepsilon\cos\theta},$$

where

$$\varepsilon = \sqrt{1 + 2{p_\theta}^2 E}.$$

(**Remark:** This is the equation of a conic section with eccentricity ε in polar coordinates).

(d) Characterize all geodesics of $\mathbb{R}^2 \setminus \{(0,0)\}$ with the Riemannian metric

$$\langle \cdot, \cdot \rangle = \frac{1}{\sqrt{x^2+y^2}} (dx \otimes dx + dy \otimes dy) .$$

Show that this manifold is isometric to the surface of a cone with aperture $\frac{\pi}{3}$.

5.2 Holonomic costrains

Exercise 2.9

(1) Use spherical coordinates to write the equations of motion for the **spherical pendulum** of length l, i.e. a particle of mass $m > 0$ moving in $\mathbb{R}^3$ subject to a constant gravitational acceleration g and the holonomic constraint

$$N = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = l^2 \right\} .$$

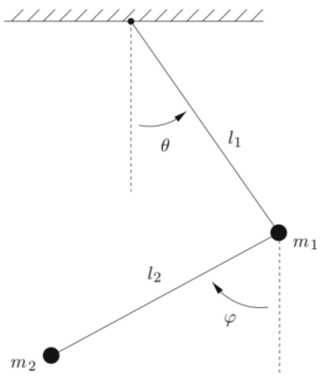
Which parallels of N are possible trajectories of the particle?

(2) Write the equations of motion for a particle moving on a frictionless surface of revolution with equation $z = f(r)$ (where $r = \sqrt{x^2 + y^2}$) under a constant gravitational acceleration g.

(3) Write and solve the equations of motion for a free dumbbell, i.e. a system of two particles of masses m_1 and m_2 connected by a massless rod of length l, moving in:

(a) $\mathbb{R}^2$;
(b) $\mathbb{R}^3$.

(**Hint:** Use the coordinates of the **center of mass**, i.e. the point along the rod at a distance $\frac{m_2}{m_1+m_2} l$ from m_1).



(4) The **double pendulum** of lengths l_1, l_2 is the mechanical system defined by two particles of masses m_1, m_2 moving in $\mathbb{R}^2$ subject to a constant gravitational acceleration g and the holonomic constraint

$$N = \left\{(x_1, x_2) \in \mathbb{R}^4 \mid \|x_1\| = l_1 \text{ and } \|x_1 - x_2\| = l_2\right\}.$$

(diffeomorphic to the 2-torus T^2).

(a) Write the equations of motion for the double pendulum using the parameterization $\phi : (-\pi, \pi) \times (-\pi, \pi) \to N$ given by

$$\phi(\theta, \varphi) = (l_1 \sin\theta, -l_1 \cos\theta, l_1 \sin\theta + l_2 \sin\varphi, -l_1 \cos\theta - l_2 \cos\varphi)$$

(cf. Fig. 5.2).

(b) Linearize the equations of motion around $\theta = \varphi = 0$. Look for solutions of the linearized equations satisfying $\varphi = k\theta$, with $k \in \mathbb{R}$ constant (**normal modes**). What are the periods of the ensuing oscillations?

5.3 Rigid Body

Exercise 3.20

(1) Show that the bilinear form $\langle\langle\cdot, \cdot\rangle\rangle$ defined on $SO(3)$ by a rigid body is indeed a Riemannian metric.

(2) A general rigid body (i.e. with no fixed points) is any mechanical system of the form $(\mathbb{R}^3 \times SO(3), \langle\langle\langle\cdot, \cdot\rangle\rangle\rangle, \mathcal{F})$, with

$$\langle\langle\langle(v, V), (w, W)\rangle\rangle\rangle := \int_{\mathbb{R}^3} \langle v + V\xi, w + W\xi\rangle \, dm$$

for all $(v, V), (w, W) \in T_{(x,S)}\mathbb{R}^3 \times SO(3)$ and $(x, S) \in \mathbb{R}^3 \times SO(3)$, where $\langle\cdot, \cdot\rangle$ is the usual Euclidean inner product on $\mathbb{R}^3$ and m is a positive finite measure on $\mathbb{R}^3$ not supported on any straight line and satisfying $\int_{\mathbb{R}^3} \|\xi\|^2 dm < +\infty$.

(a) Show that one can always translate m in such a way that

$$\int_{\mathbb{R}^3} \xi \, dm = 0$$

(i.e. the center of mass of the reference configuration is placed at the origin).

(b) Show that for this choice the kinetic energy of the rigid body is

$$K(v, V) = \frac{1}{2}M\langle v, v\rangle + \frac{1}{2}\langle\langle V, V\rangle\rangle,$$

where $M = m(\mathbb{R}^3)$ is the total mass of the rigid body and $\langle\langle\cdot, \cdot\rangle\rangle$ is the metric for the rigid body (with a fixed point) determined by m.

(c) Assume that there exists a differentiable function $F : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathcal{F}(x, S, v, V)(w, W) = \int_{\mathbb{R}^3} \langle F(x + S\xi), w + W\xi\rangle \, dm.$$

Show that, if

$$\int_{\mathbb{R}^3} (S\xi) \times F(x + S\xi) \, dm = 0$$

for all $(x, S) \in \mathbb{R}^3 \times SO(3)$, then the projection of any motion on $SO(3)$ is a geodesic of $(SO(3), \langle\langle\cdot, \cdot\rangle\rangle)$.

(d) Describe the motion of a rigid body falling in a constant gravitational field, for which $F = -ge_z$ is constant.

(3) Prove Proposition 3.6 for a planar rigid body. (**Hint:** Include the planar rigid body in a smooth one-parameter family of non-planar rigid bodies).

(4) Prove Lemma 3.9.

(5) Show that $I_1 \leq I_2 + I_3$ (and cyclic permutations). When is $I_1 = I_2 + I_3$?

(6) Determine the principal axes and the corresponding principal moments of inertia of:

(a) a homogeneous rectangular parallelepiped with mass M, sides $2a, 2b, 2c \in \mathbb{R}^+$ and centered at the origin;
(b) a homogeneous (solid) ellipsoid with mass M, semiaxes $a, b, c \in \mathbb{R}^+$ and centered at the origin. (**Hint:** Use the coordinate change $(x, y, z) = (au, bv, cw)$).

(7) A **symmetry** of a rigid body is an isometry $S \in O(3)$ which preserves the mass distribution (i.e. $m(SA) = m(A)$ for any measurable set $A \subset \mathbb{R}^3$). Show that:

(a) $SIS^t = I$, where I is the matrix representation of the inertia tensor;
(b) if S is a reflection in a plane then there exists a principal axis orthogonal to the reflection plane;
(c) if S is a nontrivial rotation about an axis then that axis is principal;
(d) if moreover the rotation is not by π then all axes orthogonal to the rotation axis are principal.

(8) Consider a rigid body satisfying $I_1 = I_2$. Use the Euler equations to show that:

(a) the angular velocity satisfies

$$\dot{\omega} = \frac{1}{I_1} p \times \omega;$$

(b) if $I_1 = I_2 = I_3$ then the rigid body rotates about a fixed axis with constant angular speed (i.e. ω is constant);
(c) if $I_1 = I_2 \neq I_3$ then ω **precesses** (i.e. rotates) about p with angular velocity

$$\omega_{\text{pr}} := \frac{p}{I_1}.$$

(9) Many asteroids have irregular shapes, and hence satisfy $I_1 < I_2 < I_3$. To a very good approximation, their rotational motion about the center of mass is described by the Euler equations. Over very long periods of time, however, their small interactions with the Sun and other planetary bodies tend to decrease their kinetic energy while conserving their angular momentum. Which rotation state do asteroids approach?

(10) Due to its rotation, the Earth is not a perfect sphere, but an oblate spheroid; therefore its moments of inertia are not quite equal, satisfying approximately

$$I_1 = I_2 \neq I_3;$$
$$\frac{I_3 - I_1}{I_1} \simeq \frac{1}{306}.$$

The Earth's rotation axis is very close to e_3, but precesses around it (**Chandler precession**). Find the period of this precession (in the Earth's frame).

(11) Consider a rigid body whose motion is described by the curve $S : \mathbb{R} \to SO(3)$, and let Ω be the corresponding angular velocity. Consider a particle with mass m whose motion **in the rigid body's frame** is given by the curve $\xi : \mathbb{R} \to \mathbb{R}^3$. Let f be the external force on the particle, so that its equation of motion is

$$m\frac{d^2}{dt^2}(S\xi) = f.$$

(a) Show that the equation of motion can be written as

$$m\ddot{\xi} = F - m\Omega \times (\Omega \times \xi) - 2m\Omega \times \dot{\xi} - m\dot{\Omega} \times \xi$$

where $f = SF$. (The terms following F are the so-called **inertial forces**, and are known, respectively, as the **centrifugal force**, the **Coriolis force** and the **Euler force**).

(b) Show that if the rigid body is a homogeneous sphere rotating freely (like the Earth, for instance) then the Euler force vanishes. Why must a long range gun in the Northern hemisphere be aimed at the **left** of the target?

(12) (*Poinsot theorem*) The **inertia ellipsoid** of a rigid body with moment of inertia tensor I is the set

$$E = \left\{ \xi \in \mathbb{R}^3 \mid \langle I\xi, \xi \rangle = 1 \right\}.$$

Show that the inertia ellipsoid of a freely moving rigid body rolls without slipping on a fixed plane orthogonal to p (that is, the contact point has zero velocity at each instant). (**Hint:** Show that any point $S(t)\xi(t)$ where the ellipsoid is tangent to a plane orthogonal to p satisfies $S(t)\xi(t) = \pm\frac{1}{\sqrt{2K}}\omega(t)$).

(13) Prove Proposition 3.19. (**Hint:** Notice that symmetry demands that the expression for K must not depend neither on φ nor on ψ).

(14) Consider the Lagrange top.

(a) Write the equations of motion and determine the equilibrium points.
(b) Show that there exist solutions such that θ, $\dot{\varphi}$ and $\dot{\psi}$ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (**fast top**) satisfy

$$\dot{\varphi} \simeq \frac{Mgl}{I_3\dot{\psi}}.$$

(15) (*Precession of the equinoxes*) Due to its rotation, the Earth is not a perfect sphere, but an oblate ellipsoid; therefore its moments of inertia are not quite equal, satisfying approximately

$$I_1 = I_2 \neq I_3;$$
$$\frac{I_3 - I_1}{I_1} \simeq \frac{1}{306}$$

[cf. Exercise 3.20(10)]. As a consequence, the combined gravitational attraction of the Moon and the Sun disturbs the Earth's rotation motion. This perturbation can be approximately modeled by the potential energy $U : SO(3) \to \mathbb{R}$ given in the Euler angles (θ, φ, ψ) by

$$U = -\frac{\Omega^2}{2}(I_3 - I_1)\cos^2\theta,$$

where $\frac{2\pi}{\Omega} \simeq 168$ days.

(a) Write the equations of motion and determine the equilibrium points.

(b) Show that there exist solutions such that θ, $\dot{\varphi}$ and ψ are constant, which in the limit $|\dot{\varphi}| \ll |\dot{\psi}|$ (as is the case with the Earth) satisfy

$$\dot{\varphi} \simeq -\frac{\Omega^2(I_3 - I_1)\cos\theta}{I_3\dot{\psi}}.$$

Given that for the Earth $\theta \simeq 23°$, determine the approximate value of the period of $\varphi(t)$.

(16) (*Pseudo-rigid body*) Recall that the (non planar) rigid body metric is the restriction to $SO(3)$ of the flat metric on $GL(3)$ given by

$$\langle\langle V, W\rangle\rangle = \operatorname{tr}\left(VJW^t\right),$$

where

$$J_{ij} = \int_{\mathbb{R}^3} \xi^i \xi^j \, dm.$$

(a) What are the geodesics of the Levi–Civita connection for this metric? Is $(GL(3), \langle\langle\cdot,\cdot\rangle\rangle)$ geodesically complete?

(b) The **Euler equation** and the **continuity equation** for an incompressible fluid with velocity field $u : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p,$$
$$\nabla \cdot u = 0,$$

where

$$\nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

is the usual operator of vector calculus.

Given a geodesic $S : \mathbb{R} \to GL(3)$, we define

$$x(t, \xi) = S(t)\xi,$$
$$u(t, x) = \dot{S}(t)\xi = \dot{S}(t)S^{-1}(t)x.$$

Show that the velocity field u satisfies the Euler equation (with $p = 0$), but not the continuity equation.

(c) Let $f : GL(3) \to \mathbb{R}$ be given by $f(S) = \det S$. Show that

$$\frac{\partial f}{\partial S_{ij}} = \operatorname{cof}(S)_{ij}$$

(where $\operatorname{cof}(S)$ is the matrix of the cofactors of S), and consequently

$$\frac{df}{dt} = (\det S) \operatorname{tr}\left(\dot{S}S^{-1}\right).$$

So the continuity equation is satisfied if we impose the constraint $\det S(t) = 1$.

(d) Show that the holonomic constraint $SL(3) \subset GL(3)$ satisfies the d'Alembert principle if and only if

$$\begin{cases} \mu\left(\ddot{S}\right) = \lambda(t) df \\ \det S = 1. \end{cases}$$

Assuming that J is invertible, show that the equation of motion can be rewritten as

$$\ddot{S} = \lambda \left(S^{-1}\right)^t J^{-1}.$$

(e) Show that the geodesics of $(SL(3), \langle\langle\cdot,\cdot\rangle\rangle)$ yield solutions of the Euler equation with

$$p = -\frac{\lambda}{2}x^t\left(S^{-1}\right)^t J^{-1}S^{-1}x$$

which also satisfy the continuity equation.

(**Remark:** More generally, it is possible to interpret the Euler equation on an open set $U \subset \mathbb{R}^n$ as a mechanical system on the group of diffeomorphisms of U (which is an infinite-dimensional Lie group); the continuity equation imposes the holonomic constraint corresponding to the subgroup of volume-preserving diffeomorphisms, and the pressure is the perfect reaction force associated to this constraint).

5.4 Non-holonomic Costrains

Exercise 4.15

(1) Show that an m-dimensional distribution Σ on an n-manifold M is differentiable if and only if for all $p \in M$ there exists a neighborhood $U \ni p$ and 1-forms $\omega^1, \ldots, \omega^{n-m} \in \Omega^1(U)$ such that

$$\Sigma_q = \ker\left(\omega^1\right)_q \cap \cdots \cap \ker\left(\omega^{n-m}\right)_q$$

for all $q \in U$.

(2) Show that the foliation

$$\mathcal{F} = \left\{(x,y) \in \mathbb{R}^2 \mid y = \sqrt{2}x + \alpha\right\}_{\alpha\in\mathbb{R}}$$

of $\mathbb{R}^2$ induces a foliation $\mathcal{F}'$ on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ whose leaves are not (embedded) submanifolds.

(3) Let Σ be an integrable distribution. Show that $X, Y \in \mathfrak{X}(\Sigma) \Rightarrow [X, Y] \in \mathfrak{X}(\Sigma)$.

(4) Using the Frobenius theorem show that an m-dimensional distribution Σ is integrable if and only if each local basis of vector fields $\{X_1, \ldots, X_m\}$ satisfies $[X_i, X_j] = \sum_{k=1}^{m} C_{ij}^k X_k$ for locally defined functions C_{ij}^k. (**Remark:** Since the condition of the Frobenius theorem is local, this condition needs to be checked only for local bases whose domains form an open cover of M).

(5) Prove Proposition 4.9. (**Hint:** Recall from Exercise 3.8(2) in Chap. 2 that $d\omega(X, Y) = X \cdot \omega(Y) - Y \cdot \omega(X) - \omega([X, Y])$ for any $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$).

(6) Let M be an n-dimensional differentiable manifold with an affine connection ∇. Show that the parallel transport of vectors is determined by a distribution Σ on TM, which is integrable if and only if the curvature of ∇ vanishes.

(7) Prove Theorem 4.14.

(8) (*Ice skate*) Recall that our model for an ice skate is given by the non-holonomic constraint Σ defined on $\mathbb{R}^2 \times S^1$ by the kernel of the 1-form $\omega = -\sin\theta dx + \cos\theta dy$.

(a) Show that the ice skate can access all points in the configuration space: given two points $p, q \in \mathbb{R}^2 \times S^1$ there exists a piecewise smooth curve $c : [0, 1] \to \mathbb{R}^2 \times S^1$ compatible with Σ such that $c(0) = p$ and $c(1) = q$. Why does this show that Σ is non-integrable?

(b) Assuming that the kinetic energy of the skate is

$$K = \frac{M}{2}\left(\left(v^x\right)^2 + \left(v^y\right)^2\right) + \frac{I}{2}\left(v^\theta\right)^2$$

and that the reaction force is perfect, show that the skate moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?

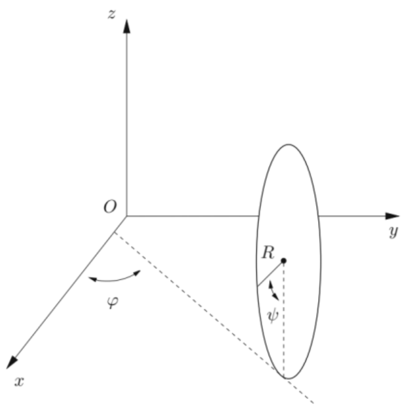
(c) Determine the motion of the skate moving on an inclined plane, i.e. subject to a potential energy $U = Mg \sin\alpha\, x$.

(9) Consider a vertical wheel of radius R moving on a plane.

(a) Show that the non-holonomic constraint corresponding to the condition of rolling without slipping or sliding is the distribution determined on the configuration space $\mathbb{R}^2 \times S^1 \times S^1$ by the 1-forms

$$\omega^1 = dx - R\cos\varphi\, d\psi, \qquad \omega^2 = dy - R\sin\varphi\, d\psi,$$

where (x, y, ψ, φ) are the local coordinates indicated in Fig. 5.9.



(b) Assuming that the kinetic energy of the wheel is

$$K = \frac{M}{2}\left((v^x)^2 + (v^y)^2\right) + \frac{I}{2}(v^\psi)^2 + \frac{J}{2}(v^\varphi)^2$$

and that the reaction force is perfect, show that the wheel moves with constant speed along straight lines or circles. What is the physical interpretation of the reaction force?

(c) Determine the motion of the vertical wheel moving on an inclined plane, i.e. subject to a potential energy $U = Mg\sin\alpha\, x$.

(10) Consider a sphere of radius R and mass M rolling without slipping on a plane.

(a) Show that the condition of rolling without slipping is

$$\dot{x} = R\omega^y, \qquad \dot{y} = -R\omega^x,$$

where (x, y) are the Cartesian coordinates of the contact point on the plane and ω is the angular velocity of the sphere.

(b) Show that if the sphere's mass is symmetrically distributed then its kinetic energy is

$$K = \frac{M}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{I}{2}\langle\omega, \omega\rangle,$$

where I is the sphere's moment of inertia and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product.

(c) Using ω as coordinates on the fibers of $TSO(3)$, show that

$$\frac{D\dot{c}}{dt} = \ddot{x}\frac{\partial}{\partial x} + \ddot{y}\frac{\partial}{\partial y} + \dot{\omega}.$$

(**Hint:** Recall from Exercise 4.8(3) in Chap. 3 that the integral curves of left-invariant vector fields on a Lie group with a bi-invariant metric are geodesics).

(d) Since we are identifying the fibers of $TSO(3)$ with $\mathbb{R}^3$, we can use the Euclidean inner product to also identify the fibers of $T^*SO(3)$ with $\mathbb{R}^3$.

Show that under this identification the non-holonomic constraint yielding the condition of rolling without slipping is the distribution determined by the kernels of the 1-forms

$$\theta^x := dx - R\,e_y, \qquad \theta^y := dy + R\,e_x$$

(where $\{e_x, e_y, e_z\}$ is the canonical basis of $\mathbb{R}^3$). Is this distribution integrable? (**Hint:** Show that any two points of $\mathbb{R}^2 \times SO(3)$ can be connected by a piecewise smooth curve compatible with the distribution).

(e) Show that the sphere moves along straight lines with constant speed and constant angular velocity orthogonal to its motion.

(f) Determine the motion of the sphere moving on an inclined plane, i.e. subject to a potential energy $U = Mg\sin\alpha\, x$.

(11) (*The golfer dilemma*) Show that the center of a symmetric sphere of radius R, mass M and moment of inertia I rolling without slipping inside a vertical cylinder of radius $R + a$ moves with constant angular velocity with respect to the axis of the cylinder while oscillating up and down with a frequency $\sqrt{\frac{I}{I+MR^2}}$ times the frequency of the angular motion.

5.5 Lagrangian Mechanics

Exercise 5.14

(1) Complete the proof of Theorem 5.3.

(2) Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold. Show that the critical points of the arclength, i.e. of the action determined by the Lagrangian $L : TM \to \mathbb{R}$ given by

$$L(v) = \langle v, v\rangle^{\frac{1}{2}}$$

(where we must restrict the action to curves with nonvanishing velocity) are reparameterized geodesics.

(3) (*Brachistochrone curve*) A particle with mass m moves on a curve $y = y(x)$ under the action of a constant gravitational field, corresponding to the potential energy $U = mgy$. The curve satisfies $y(0) = y(d) = 0$ and $y(x) < 0$ for $0 < x < d$.

(a) Assuming that the particle is set free at the origin with zero velocity, show that its speed at each point is

$$v = \sqrt{-2gy},$$

and that therefore the travel time between the origin and point $(d, 0)$ is

$$S = (2g)^{-\frac{1}{2}} \int_0^d \left(1 + y'^2\right)^{\frac{1}{2}} (-y)^{-\frac{1}{2}} dx,$$

where $y' = \frac{dy}{dx}$.

(b) Show that the curve $y = y(x)$ which corresponds to the minimum travel time satisfies the differential equation

$$\frac{d}{dx}\left[\left(1 + y'^2\right) y\right] = 0.$$

(c) Check that the solution of this equation satisfying $y(0) = y(d) = 0$ is given parametrically by

$$\begin{cases} x = R\theta - R\sin\theta \\ y = -R + R\cos\theta \end{cases}$$

where $d = 2\pi R$. (**Remark:** This curve is called a **cycloid**, because it is the curved traced out by a point on a circle which rolls without slipping on the xx-axis).

(4) (*Charged particle in a stationary electromagnetic field*) The motion of a particle with mass $m > 0$ and charge $e \in \mathbb{R}$ in a stationary electromagnetic field is determined by the Lagrangian $L : T\mathbb{R}^3 \to \mathbb{R}$ given by

$$L = \frac{1}{2}m\langle v, v\rangle + e\langle A, v\rangle - e\,\Phi,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product, $\Phi \in C^\infty(\mathbb{R}^3)$ is the **electric potential** and $A \in \mathfrak{X}(\mathbb{R}^3)$ is the **magnetic vector potential**.

(a) Show that the equations of motion are

$$m\ddot{x} = eE + e\dot{x} \times B,$$

where $E = -\operatorname{grad}\Phi$ is the **electric field** and $B = \operatorname{curl} A$ is the **magnetic field**.

(b) Write an expression for the Hamiltonian function and use the equations of motion to check that it is constant along any motion.

(5) (*Restricted 3-body problem*) Consider two gravitating particles moving in circular orbit around their common center of mass. We choose our units so that the masses of the particles are $0 < \mu < 1$ and $1 - \mu$, the distance between them is 1 and the orbital angular velocity is also 1. Identifying the plane of the orbit with $\mathbb{R}^2$, with the center of mass at the origin, we can choose fixed positions $p_1 = (1 - \mu, 0)$ and $p_2 = (-\mu, 0)$ for the particles in the rotating frame where they are at rest.

(a) Use Exercise 3.20(11) to show that in this frame the equations of motion of a third particle with negligible mass m moving in the plane of the orbit are

$$\begin{cases} \ddot{x} = \dfrac{F_x}{m} + x + 2\dot{y} \\ \\ \ddot{y} = \dfrac{F_y}{m} + y - 2\dot{x} \end{cases},$$

where $\left(F_x, F_y\right)$ is the force on m as measured in the rotating frame.

(b) Assume that the only forces on m are the gravitational forces produced by μ and $1-\mu$, so that

$$\begin{cases} \dfrac{F_x}{m} = -\dfrac{\mu}{{r_1}^3}(x-1+\mu) - \dfrac{1-\mu}{{r_2}^3}(x+\mu) \\[2ex] \dfrac{F_y}{m} = -\dfrac{\mu}{{r_1}^3}y - \dfrac{1-\mu}{{r_2}^3}y = 0 \end{cases},$$

where $r_1, r_2 : \mathbb{R}^2 \to \mathbb{R}$ are the Euclidean distances to p_1, p_2. Show that the equations of motion are the Euler–Lagrange equations for the Lagrangian $L : T\left(\mathbb{R}^2 \setminus \{p_1, p_2\}\right) \to \mathbb{R}$ given by

$$\begin{aligned} L\left(x, y, v^x, v^y\right) &= \frac{1}{2}\left(\left(v^x\right)^2 + \left(v^y\right)^2\right) + x v^y - y v^x \\ &+ \frac{1}{2}\left(x^2 + y^2\right) + \frac{\mu}{r_1} + \frac{1-\mu}{r_2}. \end{aligned}$$

(c) Find the Hamiltonian function. (**Remark:** The fact that this function remains constant gives the so-called **Tisserand criterion** for identifying the same comet before and after a close encounter with Jupiter).

(d) Compute the equilibrium points (i.e. the points corresponding to stationary solutions) which are not on the x-axis. How many equilibrium points are there in the x-axis?

(e) Show that the linearization of the system around the equilibrium points not in the x-axis is

$$\begin{cases} \ddot{\xi} - 2\dot{\eta} = \dfrac{3}{4}\xi \pm \dfrac{3\sqrt{3}}{4}(1-2\mu)\eta \\[2ex] \ddot{\eta} + 2\dot{\xi} = \pm\dfrac{3\sqrt{3}}{4}(1-2\mu)\xi + \dfrac{9}{4}\eta \end{cases},$$

and show that these equilibrium points are unstable for

$$\frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right) < \mu < \frac{1}{2}\left(1 + \frac{\sqrt{69}}{9}\right).$$

(6) Consider the mechanical system in Example 5.13.

(a) Use the Noether theorem to prove that the **total linear momentum**

$$P := \sum_{i=1}^{k} m_i \dot{x}_i$$

is conserved along the motion.

(b) Show that the system's **center of mass**, defined as the point

$$X = \frac{\sum_{i=1}^{k} m_i x_i}{\sum_{i=1}^{k} m_i},$$

moves with constant velocity.

(7) Generalize Example 5.13 to the case in which the particles move in an arbitrary Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, by showing that given any Killing vector field $X \in \mathfrak{X}(M)$ [cf. Exercise 3.3(8) in Chap. 3] the quantity

$$J^X = \sum_{i=1}^{k} m_i \langle \dot{c}_i, X \rangle$$

is conserved, where $c_i : I \subset \mathbb{R} \to M$ is the motion of the particle with mass m_i.

(8) Consider the action of $SO(3)$ on itself by left multiplication.

(a) Show that the infinitesimal action of $B \in \mathfrak{so}(3)$ is the **right**-invariant vector field determined by B.
(b) Use the Noether theorem to show that the angular momentum of the free rigid body is constant.

(9) Consider a satellite equipped with a small rotor, i.e. a cylinder which can spin freely about its axis. When the rotor is locked the satellite can be modeled by a free rigid body with inertia tensor I. The rotor's axis passes through the satellite's

center of mass, and its direction is given by the unit vector e. The rotor's mass is symmetrically distributed around the axis, producing a moment of inertia J.

(a) Show that the configuration space for the satellite with unlocked rotor is the Lie group $SO(3) \times S^1$, and that its motion is a geodesic of the left-invariant metric corresponding to the kinetic energy

$$K = \frac{1}{2}\langle I\Omega, \Omega\rangle + \frac{1}{2}J\varpi^2 + J\varpi\langle\Omega, e\rangle,$$

where the $\Omega \in \mathbb{R}^3$ is the satellite's angular velocity as seen on the satellite's frame and $\varpi \in \mathbb{R}$ is the rotor's angular speed around its axis.

(b) Use the Noether theorem to show that $l = J(\varpi + \langle\Omega, e\rangle) \in \mathbb{R}$ and $p = S(I\Omega + J\varpi e) \in \mathbb{R}^3$ are conserved along the motion of the satellite with unlocked rotor, where $S : \mathbb{R} \to SO(3)$ describes the satellite's orientation.

5.6 Hamiltonian Mechanics

Exercise 6.15

(1) Prove Proposition 6.5.

(2) Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, $\alpha \in \Omega^1(M)$ a 1-form and $U \in C^\infty(M)$ a differentiable function.

(a) Show that the Euler–Lagrange equations for the Lagrangian $L : TM \to \mathbb{R}$ given by

$$L(v) = \frac{1}{2}\langle v, v\rangle + \iota(v)\alpha_p - U(p)$$

for $v \in T_pM$ yield the motions of the mechanical system $(M, \langle\cdot,\cdot\rangle, \mathcal{F})$, where

$$\mathcal{F}(v) = -(dU)_p - \iota(v)(d\alpha)_p$$

for $v \in T_pM$.

(b) Show that the mechanical energy $E = K + U$ is conserved along the motions of $(M, \langle\cdot,\cdot\rangle, \mathcal{F})$ (which is therefore called a **conservative mechanical system with magnetic term**).

(c) Show that L is hyper-regular and compute the Legendre transformation.

(d) Find the Hamiltonian $H : T^*M \to \mathbb{R}$ and write the Hamilton equations.

(3) Let $c > 0$ be a positive number, representing the speed of light, and consider the open set $U := \{v \in T\mathbb{R}^n \mid \|v\| < c\}$, where $\|\cdot\|$ is the Euclidean norm. The motion of a free relativistic particle of mass $m > 0$ is determined by the Lagrangian $L : U \to \mathbb{R}$ given by

$$L(v) := -mc^2\sqrt{1 - \frac{\|v\|^2}{c^2}}.$$

(a) Show that L is hyper-regular and compute the Legendre transformation.

(b) Find the Hamiltonian $H : T^*\mathbb{R}^n \to \mathbb{R}$ and write the Hamilton equations.

(4) Show that in the Poincaré recurrence theorem the set of points $\alpha \in U$ such that $\psi_t(\alpha) \in U$ for some $t \geq T$ is dense in U. (**Remark:** It can be shown that this set has full measure).

(5) Let $(M, \langle\cdot,\cdot\rangle)$ be a compact Riemannian manifold. Show that for each normal ball $B \subset M$ and each $T > 0$ there exist geodesics $c : \mathbb{R} \to M$ with $\|\dot{c}(t)\| = 1$ such that $c(0) \in B$ and $c(t) \in B$ for some $t \geq T$.

(6) Let $\left(x^1, \ldots, x^n, p_1, \ldots, p_n\right)$ be the usual local coordinates on T^*M. Compute X_{x^i}, X_{p_i}, $\left\{x^i, x^j\right\}$, $\left\{p_i, p_j\right\}$ and $\left\{p_i, x^j\right\}$.

(7) Show that the Poisson bracket satisfies the **Leibniz rule**

$$\{F, GH\} = \{F, G\}H + \{F, H\}G$$

for all $F, G, H \in C^\infty(T^*M)$.

5.7 Completely Integrable Systems

Exercise 7.17

(1) Show that if $F, G \in C^\infty(T^*M)$ are first integrals, then $\{F, G\}$ is also a first integral.

(2) Prove Proposition 7.3.

(3) Consider a surface of revolution $M \subset \mathbb{R}^3$ given in cylindrical coordinates (r, θ, z) by

$$r = f(z),$$

where $f : (a, b) \to (0, +\infty)$ is differentiable.

(a) Show that the geodesics of M are the critical points of the action determined by the Lagrangian $L : TM \to \mathbb{R}$ given in local coordinates by

$$L\left(\theta, z, v^\theta, v^z\right) = \frac{1}{2}\left((f(z))^2\left(v^\theta\right)^2 + \left((f'(z))^2 + 1\right)(v^z)^2\right).$$

(b) Show that the curves given in local coordinates by $\theta =$ constant or $f'(z) = 0$ are images of geodesics.

(c) Compute the Legendre transformation, show that L is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*M \to \mathbb{R}$.

(d) Show that H is completely integrable.

(e) Show that the projection on M of the invariant set

$$L_{(E,l)} := H^{-1}(E) \cap {p_\theta}^{-1}(l)$$

$(E, l > 0)$ is given in local coordinates by

$$f(z) \geq \frac{l}{\sqrt{2E}}.$$

Use this fact to conclude that if f has a strict local maximum at $z = z_0$ then the geodesic whose image is $z = z_0$ is **stable**, i.e. geodesics with initial condition close to the point in TM with coordinates $(\theta_0, z_0, 1, 0)$ stay close to the curve $z = z_0$.

(4) Recall from Example 7.5 that a particle of mass $m > 0$ moving in a central field is described by the completely integrable Hamiltonian function

$$H\left(r, \theta, p_r, p_\theta\right) = \frac{{p_r}^2}{2m} + \frac{{p_\theta}^2}{2mr^2} + u(r).$$

(a) Show that there exist circular orbits of radius r_0 whenever $u'(r_0) \geq 0$.

(b) Verify that the set of points where dH and dp_θ are not independent is the union of these circular orbits.

(c) Show that the projection of the invariant set

$$L_{(E,l)} := H^{-1}(E) \cap {p_\theta}^{-1}(l)$$

on $\mathbb{R}^2$ is given in local coordinates by

$$u(r) + \frac{l^2}{2mr^2} \leq E.$$

(d) Conclude that if $u'(r_0) \geq 0$ and

$$u''(r_0) + \frac{3u'(r_0)}{r_0} > 0$$

then the circular orbit of radius r_0 is stable.

(5) In general relativity, the motion of a particle in the gravitational field of a point mass $M > 0$ is given by the Lagrangian $L : TU \to \mathbb{R}$ written in cylindrical coordinates (u, r, θ) as

$$L = -\frac{1}{2}\left(1 - \frac{2M}{r}\right)(v^u)^2 + \frac{1}{2}\left(1 - \frac{2M}{r}\right)^{-1}(v^r)^2 + \frac{1}{2}r^2(v^\theta)^2,$$

where $U \subset \mathbb{R}^3$ is the open set given by $r > 2M$ (the coordinate u is called the **time coordinate**, and in general is different from the **proper time** of the particle, i.e. the parameter t of the curve).

(a) Show that L is hyper-regular and compute the corresponding Hamiltonian $H : T^*U \to \mathbb{R}$.

(b) Show that H is completely integrable.

(c) Show that there exist circular orbits of any radius $r_0 > 2M$, with $H < 0$ for $r_0 > 3M$, $H = 0$ for $r_0 = 3M$ and $H > 0$ for $r_0 < 3M$. (**Remark:** The orbits with $H > 0$ are not physical, since they correspond to speeds greater than the speed of light; the orbits with $H = 0$ can only be achieved by massless particles, which move at the speed of light).

(d) Show that the set of points where dH, dp_u and dp_θ are not independent (and $p_u \neq 0$) is the union of these circular orbits.

(e) Show that the projection of the invariant cylinder

$$L_{(E,k,l)} := H^{-1}(E) \cap {p_u}^{-1}(k) \cap {p_\theta}^{-1}(l)$$

on U is given in local coordinates by

$$\frac{l^2}{r^2} - \left(1 - \frac{2M}{r}\right)^{-1} k^2 \leq 2E.$$

(f) Conclude that if $r_0 > 6M$ then the circular orbit of radius r_0 is stable.

(6) Recall that the Lagrange top is the mechanical system determined by the Lagrangian $L : TSO(3) \to \mathbb{R}$ given in local coordinates by

$$L = \frac{I_1}{2}\left(\left(v^\theta\right)^2 + \left(v^\varphi\right)^2 \sin^2\theta\right) + \frac{I_3}{2}\left(v^\psi + v^\varphi \cos\theta\right)^2 - Mgl\cos\theta,$$

where (θ, φ, ψ) are the Euler angles, M is the top's mass and l is the distance from the fixed point to the center of mass.

(a) Compute the Legendre transformation, show that L is hyper-regular and write an expression in local coordinates for the Hamiltonian $H : T^*SO(3) \to \mathbb{R}$.
(b) Prove that H is completely integrable.
(c) Show that the solutions found in Exercise 3.20(14) are stable for $|\dot{\varphi}| \ll |\dot{\psi}|$ if $|\dot{\psi}|$ is large enough.

(7) Show that the the Euler top with $I_1 < I_2 < I_3$ defines a completely integrable Hamiltonian on $T^*SO(3)$.

(8) Consider the sequence formed by the first digit of the decimal expansion of each of the integers 2^n for $n \in \mathbb{N}_0$:

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5, \ldots$$

The purpose of this exercise is to answer the following question: is there a 7 in this sequence?

(a) Show that if $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then

$$\lim_{n \to +\infty} \frac{1}{n+1} \sum_{k=0}^{n} e^{2\pi i \nu k} = 0.$$

(b) Prove the following discrete version of the Birkhoff ergodicity theorem: if a differentiable function $f : \mathbb{R} \to \mathbb{R}$ is periodic with period 1 and $\nu \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in \mathbb{R}$

$$\lim_{n\to+\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x + \nu k) = \int_0^1 f(x)dx.$$

(c) Show that $\log 2$ is an irrational multiple of $\log 10$.

(d) Is there a 7 in the sequence above?

5.8 Symmetry and Reduction

Exercise 8.23

(1) Consider the symplectic structure on

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

determined by the usual volume form. Compute the Hamiltonian flow generated by the function $H(x, y, z) = z$.

(2) Let (M, ω) be a symplectic manifold. Show that:

(a) $\omega = \sum_{i=1}^{n} dp_i \wedge dx^i$ if and only if $\{x^i, x^j\} = \{p_i, p_j\} = 0$ and $\{p_i, x^j\} = \delta_{ij}$ for $i, j = 1, \ldots, n$;
(b) M is orientable;
(c) if M is compact then ω cannot be exact. (**Remark:** In particular if M is compact and all closed 2-forms on M are exact then M does not admit a symplectic structure; this is the case for all even-dimensional spheres S^{2n} with $n > 1$).

(3) Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold, $\alpha \in \Omega^1(M)$ a 1-form and $U \in C^\infty(M)$ a differentiable function.

(a) Show that $\widetilde{\omega} := \omega + \pi^* d\alpha$ is a symplectic form on T^*M, where ω is the canonical symplectic form and $\pi : T^*M \to M$ is the natural projection ($\widetilde{\omega}$ is called a **canonical symplectic form with magnetic term**).
(b) Show that the Hamiltonian flow generated by a function $\widetilde{H} \in C^\infty(T^*M)$ with respect to the symplectic form $\widetilde{\omega}$ is given by the equations

$$\begin{cases} \dot{x}^i = \dfrac{\partial \widetilde{H}}{\partial p_i} \\ \dot{p}_i = -\dfrac{\partial \widetilde{H}}{\partial x^i} + \displaystyle\sum_{j=1}^{n} \left(\dfrac{\partial \alpha_j}{\partial x^i} - \dfrac{\partial \alpha_i}{\partial x^j} \right) \dot{x}^j \end{cases} .$$

(c) The map $F : T^*M \to T^*M$ given by

$$F(\xi) := \xi - \alpha_p$$

for $\xi \in T_p^*M$ is a fiber-preserving diffeomorphism. Show that F carries the Hamiltonian flow defined in Exercise 6.15(2) to the Hamiltonian flow of $\widetilde{H}$ with respect to the symplectic form $\widetilde{\omega}$, where

$$\widetilde{H}(\xi) := \frac{1}{2}\langle \xi, \xi \rangle + U(p)$$

for $\xi \in T_p^*M$. (**Remark:** Since the projections of the two flows on M coincide, we see that the magnetic term can be introduced by changing either the Lagrangian or the symplectic form).

(4) Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold, B the Poisson bivector and $(x^1, \ldots, x^n)$ local coordinates on M. Show that:

(a) B can be written in these local coordinates as

$$B = \sum_{i,j=1}^{n} B^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

where $B^{ij} = \{x^i, x^j\}$ for $i, j = 1, \ldots, n$;

(b) the Hamiltonian vector field generated by $F \in C^\infty(M)$ can be written as

$$X_F = \sum_{i,j=1}^{p} B^{ij} \frac{\partial F}{\partial x^i} \frac{\partial}{\partial x^j};$$

(d) if $\{\cdot, \cdot\}$ arises from a symplectic form ω then $(B^{ij}) = -(\omega_{ij})^{-1}$;

(e) if B is nondegenerate then it arises from a symplectic form.

(c) the components of B must satisfy

$$\sum_{l=1}^{n}\left(B^{il}\frac{\partial B^{jk}}{\partial x^{l}}+B^{jl}\frac{\partial B^{ki}}{\partial x^{l}}+B^{kl}\frac{\partial B^{ij}}{\partial x^{l}}\right)=0$$

for all $i, j, k = 1, \ldots, n$;

以下暫略