An introduction to Riemannian Geometry Jose Natari

Ch4 Curvature

4.1 Curvature

Exercise 1.12

(1) (a) Show that the curvature operator satisfies
(i) $R(fX_1 + gX_2, Y)Z = fR(X_1, Y)Z + gR(X_2, Y)Z$;
(ii) $R(X, fY_1 + gY_2)Z = fR(X, Y_1)Z + gR(X, Y_2)Z$;
(iii) $R(X, Y)(fZ_1 + gZ_2) = fR(X, Y)Z_1 + gR(X, Y)Z_2$,
for all vector fields $X, X_1, X_2, Y, Y_1, Y_2, Z, Z_1, Z_2 \in \mathfrak{X}(M)$ and smooth functions $f, g \in C^\infty(M)$.
(b) Show that $(R(X, Y)Z)_p \in T_pM$ depends only on X_p, Y_p, Z_p. Conclude that R defines a $(3, 1)$-tensor. (**Hint:** Choose local coordinates around $p \in M$).

(2) Let (M, g) be an n-dimensional Riemannian manifold and $p \in M$. Show that if $(x^1, \ldots, x^n)$ are normal coordinates centered at p [cf. Exercise 4.8(2) in Chap. 3] then

$$R_{ijkl}(p) = \frac{1}{2}\left(\frac{\partial^2 g_{jl}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{jk}}{\partial x^i \partial x^l} + \frac{\partial^2 g_{ik}}{\partial x^j \partial x^l}\right)(p).$$

(3) Recall that if G is a Lie group endowed with a bi-invariant Riemannian metric, ∇ is the Levi–Civita connection and X, Y are two left-invariant vector fields then

$$\nabla_X Y = \frac{1}{2}[X, Y]$$

[cf. Exercise 4.8(3) in Chap. 3]. Show that if Z is also left-invariant, then

$$R(X, Y)Z = \frac{1}{4}[Z, [X, Y]].$$

(4) Show that $\|X_p\|^2 \|Y_p\|^2 - \langle X_p, Y_p \rangle^2$ gives us the square of the area of the parallelogram in T_pM spanned by X_p, Y_p. Conclude that the sectional curvature does not depend on the choice of the linearly independent vectors X_p, Y_p, that is, when we change the basis on Π, both $R(X_p, Y_p, X_p, Y_p)$ and $\|X_p\|^2 \|Y_p\|^2 - \langle X_p, Y_p \rangle^2$ change by the square of the determinant of the change of basis matrix.

(5) Show that Ric is the only independent contraction of the curvature tensor: choosing any other two indices and contracting, one either gets $\pm Ric$ or 0.

(6) Let M be a 3-dimensional Riemannian manifold. Show that the curvature tensor is entirely determined by the Ricci tensor.

(7) Let (M, g) be an n-dimensional isotropic Riemannian manifold with sectional curvature K. Show that $Ric = (n-1)Kg$ and $S = n(n-1)K$.

(8) Let g_1, g_2 be two Riemannian metrics on a manifold M such that $g_1 = \rho g_2$, for some constant $\rho > 0$. Show that:

(a) the corresponding sectional curvatures K_1 and K_2 satisfy $K_1(\Pi) = \rho^{-1}K_2(\Pi)$ for any 2-dimensional section of a tangent space of M;
(b) the corresponding Ricci curvature tensors satisfy $Ric_1 = Ric_2$;
(c) the corresponding scalar curvatures satisfy $S_1 = \rho^{-1}S_2$.

(9) If ∇ is not the Levi–Civita connection can we still define the Ricci curvature tensor Ric? Is it necessarily symmetric?

4.2 Cartan Structure Equations

Exercise 2.8

(1) Let $\{X_1, \ldots, X_n\}$ be a field of frames on an open set V of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ with Levi–Civita connection ∇. The associated **structure functions** C_{ij}^k are defined by

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k.$$

Show that:

(a) $C_{jk}^i = \Gamma_{jk}^i - \Gamma_{kj}^i$;
(b) $\Gamma_{jk}^i = \frac{1}{2}\sum_{l=1}^n g^{il}\left(X_j \cdot g_{kl} + X_k \cdot g_{jl} - X_l \cdot g_{jk}\right)$
$+\frac{1}{2}C_{jk}^i - \frac{1}{2}\sum_{l,m=1}^n g^{il}\left(g_{jm}C_{kl}^m + g_{km}C_{jl}^m\right)$;
(c) $d\omega^i + \frac{1}{2}\sum_{j,k=1}^n C_{jk}^i \omega^j \wedge \omega^k = 0$, where $\{\omega^1, \ldots, \omega^n\}$ is the field of dual coframes.

(2) Let $\{X_1, \ldots, X_n\}$ be a field of frames on an open set V of a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$. Show that a connection ∇ on M is compatible with the metric on V if and only if

$$X_k \cdot \langle X_i, X_j\rangle = \langle \nabla_{X_k} X_i, X_j\rangle + \langle X_i, \nabla_{X_k} X_j\rangle$$

for all i, j, k.

(3) Compute the Gauss curvature of:

(a) the sphere S^2 with the standard metric;

(b) the hyperbolic plane, i.e. the upper half-plane

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the metric

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

[cf. Exercise 3.3(5) of Chap. 3].

(4) Determine all surfaces of revolution with constant Gauss curvature.

(5) Let M be the image of the parameterization $\varphi : (0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\varphi(u, v) = (u \cos v, u \sin v, v),$$

and let N be the image of the parameterization $\psi : (0, +\infty) \times \mathbb{R} \to \mathbb{R}^3$ given by

$$\psi(u, v) = (u \cos v, u \sin v, \log u).$$

Consider in both M and N the Riemannian metric induced by the Euclidean metric of $\mathbb{R}^3$. Show that the map $f : M \to N$ defined by

$$f(\varphi(u, v)) = \psi(u, v)$$

preserves the Gauss curvature but is not a local isometry.

(6) Consider the metric

$$g = A^2(r) dr \otimes dr + r^2 d\theta \otimes d\theta + r^2 \sin^2\theta \, d\varphi \otimes d\varphi$$

on $M = I \times S^2$, where r is a local coordinate on $I \subset \mathbb{R}$ and (θ, φ) are spherical local coordinates on S^2.

(a) Compute the Ricci tensor and the scalar curvature of this metric.

(b) What happens when $A(r) = (1 - r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to S^3)?

(c) And when $A(r) = (1 + r^2)^{-\frac{1}{2}}$ (that is, when M is locally isometric to the **hyperbolic** 3-**space**)?

(d) For which functions $A(r)$ is the scalar curvature constant?

(7) Let M be an oriented Riemannian 2-manifold and let p be a point in M. Let D be a neighborhood of p in M homeomorphic to a disc, with a smooth boundary ∂D. Consider a point $q \in \partial D$ and a unit vector $X_q \in T_q M$. Let X be the parallel transport of X_q along ∂D in the positive direction. When X returns to q it makes an angle $\Delta\theta$ with the initial vector X_q. Using fields of positively oriented orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ such that $F_1 = X$, show that

$$\Delta\theta = \int_D K.$$

Conclude that the Gauss curvature of M at p satisfies

$$K(p) = \lim_{D \to p} \frac{\Delta\theta}{\mathrm{vol}(D)}.$$

(8) Compute the geodesic curvature of a positively oriented circle on:

(a) $\mathbb{R}^2$ with the Euclidean metric and the usual orientation;
(b) S^2 with the usual metric and orientation.

(9) Let c be a smooth curve on an oriented 2-manifold M as in the definition of geodesic curvature. Let X be a vector field parallel along c and let θ be the angle between X and $\dot{c}(s)$ along c in the given orientation. Show that the geodesic curvature of c, k_g, is equal to $\frac{d\theta}{ds}$. (**Hint:** Consider two fields of orthonormal frames $\{E_1, E_2\}$ and $\{F_1, F_2\}$ positively oriented such that $E_1 = \frac{X}{\|X\|}$ and $F_1 = \dot{c}$).

4.3 Gauss-Bonnet Theorem

Exercise 3.6

(1) Show that if $\langle \cdot, \cdot \rangle_0$, $\langle \cdot, \cdot \rangle_1$ are two Riemannian metrics on M then

$$\langle \cdot, \cdot \rangle_t := (1 - t)\langle \cdot, \cdot \rangle_0 + t\langle \cdot, \cdot \rangle_1$$

is also a Riemannian metric on M, and that the index $I_p(t)$ computed using the metric $\langle \cdot, \cdot \rangle_t$ is a continuous function of t.

(2) (*Gauss–Bonnet theorem for non-orientable manifolds*) Let (M, g) be a compact, non-orientable, 2-dimensional Riemannian manifold and let $\pi : \overline{M} \to M$ be its orientable double covering [cf. Exercise 8.6(9) in Chap. 1]. Show that:

(a) $\chi(\overline{M}) = 2\chi(M)$;

(b) $\overline{K} = \pi^* K$, where $\overline{K}$ is the Gauss curvature of the Riemannian metric $\overline{g} := \pi^* g$ on $\overline{M}$;

(c) $2\pi\chi(M) = \frac{1}{2}\int_{\overline{M}} \overline{K}$.

(**Remark:** Even though M is not orientable, we can still define the integral of a function f on M through $\int_M f = \frac{1}{2}\int_{\overline{M}} \pi^* f$; with this definition, the Gauss–Bonnet theorem holds for non-orientable Riemannian 2-manifolds).

(3) (*Gauss–Bonnet theorem for manifolds with boundary*) Let M be a compact, oriented, 2-dimensional manifold with boundary and let X be a vector field in M **transverse** to ∂M (i.e. such that $X_p \notin T_p\partial M$ for all $p \in \partial M$), with isolated singularities $p_1, \ldots, p_k \in M \setminus \partial M$. Prove that

$$\int_M K + \int_{\partial M} k_g = 2\pi \sum_{i=1}^{k} I_{p_i}$$

for any Riemannian metric on M, where K is the Gauss curvature of M and k_g is the geodesic curvature of ∂M.

(4) Let (M, g) be a compact orientable 2-dimensional Riemannian manifold, with positive Gauss curvature. Show that any two non-self-intersecting closed geodesics must intersect each other.

(5) Let M be a differentiable manifold and $f : M \to \mathbb{R}$ a smooth function.

(a) (*Hessian*) Let $p \in M$ be a critical point of f (i.e. $(df)_p = 0$). The **Hessian** of f at p is the map $(Hf)_p : T_pM \times T_pM \to \mathbb{R}$ given by

$$(Hf)_p(v, w) = \frac{\partial^2}{\partial t \partial s}_{|s=t=0} (f \circ \gamma)(s, t),$$

where $\gamma : U \subset \mathbb{R}^2 \to M$ is such that $\gamma(0, 0) = p$, $\frac{\partial \gamma}{\partial s}(0, 0) = v$ and $\frac{\partial \gamma}{\partial t}(0, 0) = w$. Show that $(Hf)_p$ is a well-defined symmetric 2-tensor.

(b) (*Morse theorem*) If $(Hf)_p$ is nondegenerate then p is called a **nondegenerate critical point**. Assume that M is compact and f is a **Morse function**, i.e. all its critical points are nondegenerate. Prove that there are only a finite number of critical points. Moreover, show that if M is 2-dimensional then

$$\chi(M) = m - s + n,$$

where m, n and s are the numbers of maxima, minima and saddle points respectively. (**Hint:** Choose a Riemannian metric on M and consider the vector field $X := \operatorname{grad} f$).

(6) Let (M, g) be a 2-dimensional Riemannian manifold and $\Delta \subset M$ a **geodesic triangle**, i.e. an open set homeomorphic to an Euclidean triangle whose sides are images of geodesic arcs. Let α, β, γ be the inner angles of Δ, i.e. the angles between the geodesics at the intersection points contained in $\partial\Delta$. Prove that for small enough Δ one has

$$\alpha + \beta + \gamma = \pi + \int_\Delta K,$$

where K is the Gauss curvature of M, using:

(a) the fact that $\int_\Delta K$ is the angle by which a vector parallel-transported once around $\partial\Delta$ rotates;
(b) the Gauss–Bonnet theorem for manifolds with boundary.
(**Remark:** We can use this result to give another geometric interpretation of the Gauss curvature: $K(p) = \lim_{\Delta \to p} \frac{\alpha+\beta+\gamma-\pi}{\operatorname{vol}(\Delta)}$).

(7) Let (M, g) be a simply connected 2-dimensional Riemannian manifold with nonpositive Gauss curvature. Show that any two geodesics intersect at most in one point. (**Hint:** Note that if two geodesics intersected in more than one point then there would exist a **geodesic biangle**, i.e. an open set homeomorphic to a disc whose boundary is formed by the images of two geodesic arcs).

4.4 Manifolds of Constant Curvature

Exercise 4.7

(1) Show that the metric of $H^n(a)$ is a left-invariant metric for the Lie group structure induced by identifying $(x^1, \ldots, x^n) \in H^n(a)$ with the affine map $g : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ given by

$$g(t^1, \ldots, t^{n-1}) = x^n(t^1, \ldots, t^{n-1}) + (x^1, \ldots, x^{n-1}).$$

(2) Prove that if the forms ω^i in a field of orthonormal coframes satisfy $d\omega^i = \alpha \wedge \omega^i$ (with α a 1-form), then the connection forms ω_i^j are given by $\omega_i^j = \alpha(E_i)\omega^j - \alpha(E_j)\omega^i = -\omega_j^i$. Use this to confirm the results in Example 4.2.

(3) Let K be a real number and let $\rho = 1 + (\frac{K}{4}) \sum_{i=1}^n (x^i)^2$. Show that, for the Riemannian metric defined on $\mathbb{R}^n$ by

$$g_{ij}(p) = \frac{1}{\rho^2} \delta_{ij},$$

the sectional curvature is constant equal to K.

(4) Show that any isometry of the Euclidean space $\mathbb{R}^n$ which preserves the coordinate function x^n is an isometry of $H^n(a)$. Use this fact to determine all the geodesics of $H^n(a)$.

(5) (*Schur theorem*) Let M be a connected isotropic Riemannian manifold of dimension $n \geq 3$. Show that M has constant curvature. (**Hint:** Use the structure equations to show that $dK = 0$).

(6) To complete the details in Example 4.4, show that:

(a) the isometries of $\mathbb{R}^2$ with no fixed points are either translations or gliding reflections;
(b) any discrete group of isometries of $\mathbb{R}^2$ acting properly and freely is generated by at most two elements, one of which may be assumed to be a translation.

(7) Let $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometries

$$f(x, y) = (-x, y + 1) \quad \text{and} \quad g(x, y) = (x + 1, y)$$

(thus f is a gliding reflection and g is a translation). Check that $\mathbb{R}^2/\langle f \rangle$ is homeomorphic to a Möbius band (without boundary), and that $\mathbb{R}^2/\langle f, g \rangle$ is homeomorphic to a Klein bottle.

(8) Let H^2 be the hyperbolic plane. Show that:

(a) the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d} \qquad (ad-bc=1)$$

defines an action of $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\{\pm \mathrm{id}\}$ on H^2 by orientation-preserving isometries;

(b) for any two geodesics $c_1, c_2 : \mathbb{R} \to H^2$, parameterized by the arclength, there exists $g \in PSL(2,\mathbb{R})$ such that $c_1(s) = g \cdot c_2(s)$ for all $s \in \mathbb{R}$;

(c) given $z_1, z_2, z_3, z_4 \in H^2$ with $d(z_1, z_2) = d(z_3, z_4)$, there exists $g \in PSL(2,\mathbb{R})$ such that $g \cdot z_1 = z_3$ and $g \cdot z_2 = z_4$;

(d) an orientation-preserving isometry of H^2 with two fixed points must be the identity. Conclude that all orientation-preserving isometries are of the form $f(z) = g \cdot z$ for some $g \in PSL(2,\mathbb{R})$.

(9) Check that the isometries $g(z) = z + 2$ and $h(z) = \frac{z}{2z+1}$ of the hyperbolic plane in Example 4.5 identify the sides of the hyperbolic polygon in Fig. 4.5.

(10) A **tractrix** is the curve described parametrically by

$$\begin{cases} x = u - \tanh u \\ y = \operatorname{sech} u \end{cases} \qquad (u > 0)$$

(its name derives from the property that the distance between any point in the curve and the x-axis along the tangent is constant equal to 1). Show that the surface of revolution generated by rotating a tractrix about the x-axis (**tractroid**) has constant Gauss curvature $K = -1$. Determine an open subset of the pseudosphere isometric to the tractroid. (**Remark:** The tractroid is not geodesically complete; in fact, it was proved by Hilbert in 1901 that any surface of constant negative curvature embedded in Euclidean 3-space must be incomplete).

(11) Show that the group of isometries of S^n is $O(n+1)$.

(12) Let G be a compact Lie group of dimension 2. Show that:

(a) G is orientable;
(b) $\chi(G) = 0$;
(c) any left-invariant metric on G has constant curvature;
(d) G is the 2-torus T^2.

4.5 Isometric Immersions

Exercise 5.7

(1) Let M be a Riemannian manifold with Levi–Civita connection $\widetilde{\nabla}$, and let N be a submanifold endowed with the induced metric and Levi–Civita connection ∇. Let $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ be local extensions of $X, Y \in \mathfrak{X}(N)$. Recall that the second fundamental form of the inclusion of N in M is the map $B : T_pN \times T_pN \to (T_pN)^\perp$ defined at each point $p \in N$ by

$$B(X, Y) := \widetilde{\nabla}_{\widetilde{X}} \widetilde{Y} - \nabla_X Y.$$

Show that:

(a) $B(X, Y)$ does not depend on the choice of the extensions $\widetilde{X}, \widetilde{Y}$;
(b) $B(X, Y)$ is orthogonal to N;
(c) B is symmetric, i.e. $B(X, Y) = B(Y, X)$;
(d) B is bilinear;
(e) $B(X, Y)_p$ depends only on the values of X_p and Y_p;
(f) $\widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{X} - \nabla_{[X,Y]} X$ is orthogonal to N.

(2) Let $S^n(r) \subset \mathbb{R}^{n+1}$ be the n dimensional sphere of radius r.

(a) Choosing at each point the outward pointing normal unit vector, what is the Gauss map of this inclusion?
(b) What are the eigenvalues of its derivative?
(c) Show that all sectional curvatures are equal to $\frac{1}{r^2}$ (so $S^n(r)$ has constant curvature $\frac{1}{r^2}$).

(3) Let $(M, \langle\cdot, \cdot\rangle)$ be a Riemannian manifold. A submanifold $N \subset M$ is said to be **totally geodesic** if the the geodesics of N are geodesics of M. Show that:

(a) N is totally geodesic if and only if $B \equiv 0$, where B is the second fundamental form of N;
(b) if N is the set of fixed points of an isometry then N is totally geodesic. Use this result to give examples of totally geodesic submanifolds of $\mathbb{R}^n$, S^n and H^n.

(4) Let N be a hypersurface in $\mathbb{R}^{n+1}$ and let p be a point in N. Show that if $K(p) \neq 0$ then

$$|K(p)| = \lim_{D \to p} \frac{\mathrm{vol}(g(D))}{\mathrm{vol}(D)},$$

where $g : V \subset N \to S^n$ is the Gauss map and D is a neighborhood of p whose diameter tends to zero.

(5) Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, p a point in M and Π a section of T_pM. For $B_\varepsilon(p) := \exp_p(B_\varepsilon(0))$ a normal ball around p consider the set $N_p := \exp_p(B_\varepsilon(0) \cap \Pi)$. Show that:

(a) the set N_p is a 2-dimensional submanifold of M formed by the segments of geodesics in $B_\varepsilon(p)$ which are tangent to Π at p;
(b) if in N_p we use the metric induced by the metric in M, the sectional curvature $K^M(\Pi)$ is equal to the Gauss curvature of the 2-manifold N_p.

(6) Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold with Levi–Civita connection $\widetilde{\nabla}$ and let N be a hypersurface in M. The **geodesic curvature** of a curve $c : I \subset \mathbb{R} \to M$, parameterized by arclength, is $k_g(s) = \|\widetilde{\nabla}_{\dot{c}(s)}\dot{c}(s)\|$. Show that the absolute values of the principal curvatures are the geodesic curvatures (in M) of the geodesics of N tangent to the principal directions. (**Remark:** In the case of an oriented 2-dimensional Riemannian manifold, k_g is taken to be positive or negative according to the orientation of $\{\dot{c}(s), \widetilde{\nabla}_{\dot{c}(s)}\dot{c}(s)\}$—cf. Sect. 4.2).

(7) Use the Gauss map to compute the Gauss curvature of the following surfaces in $\mathbb{R}^3$:

(a) the paraboloid $z = \frac{1}{2}\left(x^2 + y^2\right)$;
(b) the saddle surface $z = xy$.

(8) (*Surfaces of revolution*) Consider the map $f : \mathbb{R} \times (0, 2\pi) \to \mathbb{R}^3$ given by

$$f(s, \theta) = (h(s)\cos\theta, h(s)\sin\theta, g(s))$$

with $h > 0$ and g smooth maps such that

$$(h'(s))^2 + (g'(s))^2 = 1.$$

The image of f is the surface of revolution S with axis Oz, obtained by rotating the curve $\alpha(s) = (h(s), g(s))$, parameterized by the arclength s, around that axis.

(a) Show that f is an immersion.

(b) Show that $f_s := (df)\left(\frac{\partial}{\partial s}\right)$ and $f_\theta := (df)(\frac{\partial}{\partial \theta})$ are orthogonal.

(c) Determine the Gauss map and compute the matrix of the second fundamental form of S associated to the frame $\{E_s, E_\theta\}$, where $E_s := f_s$ and $E_\theta := \frac{1}{\|f_\theta\|} f_\theta$.

(d) Compute the mean curvature H and the Gauss curvature K of S.

(e) Using these results, give examples of surfaces of revolution with:
(1) $K \equiv 0$;
(2) $K \equiv 1$;
(3) $K \equiv -1$;

(4) $H \equiv 0$ (not a plane).
(**Remark:** Surfaces with constant zero mean curvature are called **minimal surfaces**; it can be proved that if a compact surface with boundary has minimum area among all surfaces with the same boundary then it must be a minimal surface).