An introduction to Riemannian Geometry Jose Natari

Ch3 Riemannian Manifolds

Exercise 1.10

(1) Let $g = \sum_{i,j=1}^{n} g_{ij}\, dx^i \otimes dx^j \in \mathcal{T}^2(T_p^*M)$. Show that:

(a) g is symmetric if and only if $g_{ij} = g_{ji}$ $(i, j = 1, \ldots, n)$;
(b) g is nondegenerate if and only if $\det(g_{ij}) \neq 0$;
(c) g is positive definite if and only if (g_{ij}) is a positive definite matrix;
(d) if g is nondegenerate, the map $\Phi_g : T_pM \to T_p^*M$ given by $\Phi_g(v)(w) = g(v, w)$ for all $v, w \in T_pM$ is a linear isomorphism;
(e) if g is positive definite then g is nondegenerate.

(2) Prove that any differentiable manifold admits a Riemannian structure without invoking the Whitney theorem. (**Hint:** Use partitions of unity).

(3) (a) Let (M, g) be a Riemannian manifold and let G be a discrete Lie group acting freely and properly on M by isometries. Show that M/G has a natural Riemannian structure (called the **quotient** structure).
(b) How would you define the **flat square metric** on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$?
(c) How would you define the **standard metric** on the real projective n-space $\mathbb{R}P^n = S^n/\mathbb{Z}_2$?

(4) Recall that given a Lie group G and $x \in G$, the left translation by x is the diffeomorphism $L_x : G \to G$ given by $L_x(y) = xy$ for all $y \in G$. A Riemannian metric g on G is said to be **left-invariant** if L_x is an isometry for all $x \in G$. Show that:

(a) $g(\cdot, \cdot) \equiv \langle \cdot, \cdot \rangle$ is left-invariant if and only if

$$\langle v, w \rangle_x = \left\langle \left(dL_{x^{-1}}\right)_x v, \left(dL_{x^{-1}}\right)_x w \right\rangle_e$$

for all $x \in G$ and $v, w \in T_xG$, where $e \in G$ is the identity and $\langle \cdot, \cdot \rangle_e$ is an inner product on the Lie algebra $\mathfrak{g} = T_eG$;
(b) the standard metric on $S^3 \cong SU(2)$ is left-invariant;
(c) the metric induced on $O(n)$ by the Euclidean metric of $\mathcal{M}_{n\times n} \cong \mathbb{R}^{n^2}$ is left-invariant.

(5) We say that a differentiable curve $\gamma : [\alpha, \beta] \to M$ is obtained from the curve $c : [a, b] \to M$ by **reparameterization** if there exists a smooth bijection $f : [\alpha, \beta] \to [a, b]$ (the reparameterization) such that $\gamma = c \circ f$. Show that if γ is obtained from c by reparameterization then $l(\gamma) = l(c)$.

(6) Let (M, g) be a Riemannian manifold and $f \in C^\infty(M)$. Show that if $a \in \mathbb{R}$ is a regular value of f then $\operatorname{grad}(f)$ is orthogonal to the submanifold $f^{-1}(a)$.

3.2 Affine Connections

Exercise 2.6

(1) (a) Show that if $X, Y \in \mathfrak{X}(M)$ coincide with $\widetilde{X}, \widetilde{Y} \in \mathfrak{X}(M)$ in some open set $W \subset M$ then $\nabla_X Y = \nabla_{\widetilde{X}} \widetilde{Y}$ on W. (**Hint:** Use bump functions with support contained on W and the properties listed in definition 2.1).

(b) Obtain the local coordinate expression (3.1) for $\nabla_X Y$.

(c) Obtain the local coordinate Eq. (3.3) for the parallel transport law.

(d) Obtain the local coordinate Eq. (3.4) for the geodesics of the connection ∇.

(2) Determine all affine connections on $\mathbb{R}^n$. Of these, determine the connections whose geodesics are straight lines $c(t) = at + b$ (with $a, b \in \mathbb{R}^n$).

(3) Let ∇ be an affine connection on M. If $\omega \in \Omega^1(M)$ and $X \in \mathfrak{X}(M)$, we define the **covariant derivative** of ω along X, $\nabla_X \omega \in \Omega^1(M)$, by

$$\nabla_X \omega(Y) = X \cdot (\omega(Y)) - \omega(\nabla_X Y)$$

for all $Y \in \mathfrak{X}(M)$.

(a) Show that this formula defines indeed a 1-form, i.e. show that $(\nabla_X \omega(Y))(p)$ is a linear function of Y_p.

(b) Show that

(i) $\nabla_{fX+gY}\omega = f\nabla_X \omega + g\nabla_Y \omega$;

(ii) $\nabla_X(\omega + \eta) = \nabla_X \omega + \nabla_X \eta$;

(iii) $\nabla_X(f\omega) = (X \cdot f)\omega + f\nabla_X \omega$

for all $X, Y \in \mathfrak{X}(M)$, $f, g \in C^\infty(M)$ and $\omega, \eta \in \Omega^1(M)$.

(c) Let $x : W \to \mathbb{R}^n$ be local coordinates on an open set $W \subset M$, and take

$$\omega = \sum_{i=1}^n \omega_i dx^i.$$

Show that

$$\nabla_X \omega = \sum_{i=1}^n \left(X \cdot \omega_i - \sum_{j,k=1}^n \Gamma^k_{ji} X^j \omega_k \right) dx^i.$$

(d) Define the covariant derivative $\nabla_X T$ for an arbitrary tensor field T in M, and write its expression in local coordinates.

3.3 Levi-Civita Connection

Exercise 3.3

(1) Show that the Koszul formula defines a connection.

(2) We introduce in $\mathbb{R}^3$, with the usual Euclidean metric $\langle \cdot, \cdot \rangle$, the connection ∇ defined in Cartesian coordinates (x^1, x^2, x^3) by

$$\Gamma^i_{jk} = \omega \varepsilon_{ijk},$$

where $\omega : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function and

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{otherwise.} \end{cases}$$

Show that:

(a) ∇ is compatible with $\langle \cdot, \cdot \rangle$;

(b) the geodesics of ∇ are straight lines;

(c) the torsion of ∇ is not zero in all points where $\omega \neq 0$ (therefore ∇ is not the Levi-Civita connection unless $\omega \equiv 0$);

(d) the parallel transport equation is

$$\dot{V}^i + \sum_{j,k=1}^{3} \omega \varepsilon_{ijk} \dot{x}^j V^k = 0 \Leftrightarrow \dot{V} + \omega(\dot{x} \times V) = 0$$

(where $\times$ is the cross product in $\mathbb{R}^3$); therefore, a vector parallel along a straight line rotates about it with angular velocity $-\omega \dot{x}$.

(3) Let (M, g) and $(N, \widetilde{g})$ be isometric Riemannian manifolds with Levi-Civita connections ∇ and $\widetilde{\nabla}$, and let $f : M \to N$ be an isometry. Show that:

(a) $f_* \nabla_X Y = \widetilde{\nabla}_{f_* X} f_* Y$ for all $X, Y \in \mathfrak{X}(M)$;
(b) if $c : I \to M$ is a geodesic then $f \circ c : I \to N$ is also a geodesic.

(4) Consider the usual local coordinates (θ, φ) in $S^2 \subset \mathbb{R}^3$ defined by the parameterization $\phi : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ given by

$$\phi(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

(a) Using these coordinates, determine the expression of the Riemannian metric induced on S^2 by the Euclidean metric of $\mathbb{R}^3$.

(b) Compute the Christoffel symbols for the Levi-Civita connection in these coordinates.

(c) Show that the equator is the image of a geodesic.

(d) Show that any rotation about an axis through the origin in $\mathbb{R}^3$ induces an isometry of S^2.

(e) Show that the images of geodesics of S^2 are great circles.

(f) Find a **geodesic triangle** (i.e. a triangle whose sides are images of geodesics) whose internal angles add up to $\frac{3\pi}{2}$.

(g) Let $c : \mathbb{R} \to S^2$ be given by $c(t) = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0)$, where $\theta_0 \in \left(0, \frac{\pi}{2}\right)$ (therefore c is not a geodesic). Let V be a vector field parallel along c such that $V(0) = \frac{\partial}{\partial\theta}$ ($\frac{\partial}{\partial\theta}$ is well defined at $(\sin\theta_0, 0, \cos\theta_0)$ by continuity). Compute the angle by which V is rotated when it returns to the initial point. (**Remark:** The angle you have computed is exactly the angle by which the oscillation plane of the **Foucault pendulum** rotates during a day in a place at latitude $\frac{\pi}{2} - \theta_0$, as it tries to remain fixed with respect to the stars on a rotating Earth).

(h) Use this result to prove that no open set $U \subset S^2$ is isometric to an open set $W \subset \mathbb{R}^2$ with the Euclidean metric.

(i) Given a geodesic $c : \mathbb{R} \to \mathbb{R}^2$ of $\mathbb{R}^2$ with the Euclidean metric and a point $p \notin c(\mathbb{R})$, there exists a unique geodesic $\widetilde{c} : \mathbb{R} \to \mathbb{R}^2$ (up to reparameterization) such that $p \in \widetilde{c}(\mathbb{R})$ and $c(\mathbb{R}) \cap \widetilde{c}(\mathbb{R}) = \varnothing$ (**parallel postulate**). Is this true in S^2?

(5) Recall that identifying each point in

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the invertible affine map $h : \mathbb{R} \to \mathbb{R}$ given by $h(t) = yt + x$ induces a Lie group structure on H [cf. Exercise 7.17(3) in Chap. 1].

(a) Show that the left-invariant metric induced by the Euclidean inner product $dx \otimes dx + dy \otimes dy$ in $\mathfrak{h} = T_{(0,1)}H$ is

$$g = \frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$$

(b) Compute the Christoffel symbols of the Levi-Civita connection in the coordinates (x, y).

(c) Show that the curves $\alpha, \beta : \mathbb{R} \to H$ given in these coordinates by

$$\alpha(t) = \left(0, e^t\right)$$
$$\beta(t) = \left(\tanh t, \frac{1}{\cosh t}\right)$$

are geodesics. What are the sets $\alpha(\mathbb{R})$ and $\beta(\mathbb{R})$?

(d) Determine all images of geodesics.

(e) Show that, given two points $p, q \in H$, there exists a unique geodesic through them (up to reparameterization).

(f) Give examples of connected Riemannian manifolds containing two points through which there are (i) infinitely many geodesics (up to reparameterization); (ii) no geodesics.

(g) Show that no open set $U \subset H$ is isometric to an open set $V \subset \mathbb{R}^2$ with the Euclidean metric. (**Hint:** Show that in any neighborhood of any point $p \in H$ there is always a geodesic quadrilateral whose internal angles add up to less than 2π).

(h) Does the parallel postulate hold in the hyperbolic plane?

(6) Let $(M, \langle\cdot,\cdot\rangle)$ be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$, and let $(N, \langle\langle\cdot,\cdot\rangle\rangle)$ be a submanifold with the induced metric and Levi-Civita connection ∇.

(a) Show that

$$\nabla_X Y = \left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\right)^\top$$

for all $X, Y \in \mathfrak{X}(N)$, where $\widetilde{X}, \widetilde{Y}$ are any extensions of X, Y to $\mathfrak{X}(M)$ and $^\top : TM|_N \to TN$ is the orthogonal projection.

(b) Use this result to indicate curves that are, and curves that are not, geodesics of the following surfaces in $\mathbb{R}^3$:
  (i) the sphere S^2;
  (ii) the torus of revolution;
  (iii) the surface of a cone;
  (iv) a general surface of revolution.

(c) Show that if two surfaces in $\mathbb{R}^3$ are tangent along a curve, then the parallel transport of vectors along this curve in both surfaces coincides.

(d) Use this result to compute the angle $\Delta\theta$ by which a vector V is rotated when it is parallel transported along a circle on the sphere. (**Hint:** Consider the cone which is tangent to the sphere along the circle (cf. Fig. 3.1); notice that the cone minus a ray through the vertex is isometric to an open set of the Euclidean plane).

(7) Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Show that g is **parallel** along any curve, i.e. show that

$$\nabla_X g = 0$$

for all $X \in \mathfrak{X}(M)$ [cf. Exercise 2.6(3)].

(8) Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇, and let $\psi_t : M \to M$ be a 1-parameter group of isometries. The vector field $X \in \mathfrak{X}(M)$ defined by

$$X_p := \frac{d}{dt}_{|t=0} \psi_t(p)$$

is called the **Killing vector field** associated to ψ_t. Show that:

(a) $L_X g = 0$ [cf. Exercise 2.8(3)];
(b) X satisfies $\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle = 0$ for all vector fields $Y, Z \in \mathfrak{X}(M)$;
(c) if $c : I \to M$ is a geodesic then $\left\langle \dot{c}(t), X_{c(t)} \right\rangle$ is constant.

(9) Recall that if M is an oriented differential manifold with volume element $\omega \in \Omega^n(M)$, the **divergence** of X is the function $\operatorname{div}(X)$ such that

$$L_X\omega = (\operatorname{div}(X))\omega$$

[cf. Exercise 6.4(5) in Chap. 2]. Suppose that M has a Riemannian metric and that ω is a Riemannian volume element.Show that at each point $p \in M$,

$$\operatorname{div}(X) = \sum_{i=1}^{n} \langle \nabla_{Y_i} X, Y_i \rangle,$$

where $\{Y_1, \ldots, Y_n\}$ is an orthonormal basis of T_pM and ∇ is the Levi-Civita connection.

3.4 Minimizing Properties of Geodsics

Exercise 4.8

(1) Let (M, g) be a Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function. Show that if $\|\operatorname{grad}(f)\| \equiv 1$ then the integral curves of $\operatorname{grad}(f)$ are geodesics, using:
   (a) the definition of geodesic;
   (b) the minimizing properties of geodesics.

(2) Let M be a Riemannian manifold and ∇ the Levi–Civita connection on M. Given $p \in M$ and a basis $\{v_1, \ldots, v_n\}$ for T_pM, we consider the parameterization $\varphi : U \subset \mathbb{R}^n \to M$ of a normal neighborhood given by

$$\varphi(x^1, \ldots, x^n) = \exp_p(x^1 v_1 + \cdots + x^n v_n)$$

(the local coordinates $(x^1, \ldots, x^n)$ are called **normal coordinates**).
Show that:
   (a) in these coordinates, $\Gamma^i_{jk}(p) = 0$ (**Hint:** Consider the geodesic equation);
   (b) if $\{v_1, \ldots, v_n\}$ is an orthonormal basis then $g_{ij}(p) = \delta_{ij}$.

(3) Let G be a Lie group endowed with a **bi-invariant Riemannian metric** (i.e. such that L_g and R_g are isometries for all $g \in G$), and let $i : G \to G$ be the diffeomorphism defined by $i(g) = g^{-1}$.
   (a) Compute $(di)_e$ and show that

$$(di)_g = \left(dR_{g^{-1}}\right)_e (di)_e \left(dL_{g^{-1}}\right)_g$$

   for all $g \in G$. Conclude that i is an isometry.

(b) Let $v \in \mathfrak{g} = T_e G$ and c_v be the geodesic satisfying $c_v(0) = e$ and $\dot{c}_v(0) = v$. Show that if t is sufficiently small then $c_v(-t) = (c_v(t))^{-1}$. Conclude that c_v is defined in $\mathbb{R}$ and satisfies $c_v(t+s) = c_v(t)c_v(s)$ for all $t, s \in \mathbb{R}$. (**Hint:** Recall that any two points in a totally normal neighborhood are connected by a unique geodesic in that neighborhood).

(c) Show that the geodesics of G are the integral curves of left-invariant vector fields, and that the maps exp (the Lie group exponential) and $\exp_e$ (the geodesic exponential at the identity) coincide.

(d) Let ∇ be the Levi-Civita connection of the bi-invariant metric and X, Y two left-invariant vector fields. Show that

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

(e) Check that the left-invariant metrics Exercise 1.10(4) are actually bi-invariant.

(f) Show that any compact Lie group admits a bi-invariant metric. (**Hint:** Take the average of a left-invariant metric over all right translations).

(4) Use Theorem 4.6 to prove that if $f : M \to N$ is an isometry and $c : I \to M$ is a geodesic then $f \circ c : I \to N$ is also a geodesic.

(5) Let $f : M \to M$ be an isometry whose set of fixed points is a connected 1-dimensional submanifold $N \subset M$. Show that N is the image of a geodesic.

(6) Let $(M, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold whose geodesics can be extended for all values of their parameters, and let $p \in M$.

(a) Let X and Y_i be the vector fields defined on a normal ball centered at p as in (3.6) and (3.7). Show that Y_i satisfies the **Jacobi equation**

$$\nabla_X \nabla_X Y_i = R(X, Y_i)X,$$

where $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

is called the **curvature operator** (cf. Chap. 4). (**Remark:** It can be shown that $(R(X, Y)Z)_p$ depends only on X_p, Y_p, Z_p).

(b) Consider a geodesic $c : \mathbb{R} \to M$ parameterized by the arclength such that $c(0) = p$. A vector field Y along c is called a **Jacobi field** if it satisfies the Jacobi equation along c,

$$\frac{D^2 Y}{dt^2} = R(\dot{c}, Y)\,\dot{c}.$$

Show that Y is a Jacobi field with $Y(0) = 0$ if and only if

$$Y(t) = \frac{\partial}{\partial s} \exp_p(tv(s))_{|s=0}$$

with $v : (-\varepsilon, \varepsilon) \to T_p M$ satisfying $v(0) = \dot{c}(0)$.

(c) A point $q \in M$ is said to be **conjugate** to p if it is a critical value of $\exp_p$. Show that q is conjugate to p if and only if there exists a nonzero Jacobi field Y along a geodesic c connecting $p = c(0)$ to $q = c(b)$ such that $Y(0) = Y(b) = 0$. Conclude that if q is conjugate to p then p is conjugate to q.

(d) The manifold M is said to have **nonpositive curvature** if $\langle R(X, Y)X, Y\rangle \geq 0$ for all $X, Y \in \mathfrak{X}(M)$. Show that for such a manifold no two points are conjugate.

(e) Given a geodesic $c : I \to M$ parameterized by the arclength such that $c(0) = p$, let t_c be the supremum of the set of values of t such that c is the minimizing curve connecting p to $c(t)$ (hence $t_c > 0$). The **cut locus** of p is defined to be the set of all points of the form $c(t_c)$ for $t_c < +\infty$. Determine the cut locus of a given point $p \in M$ when M is:
(i) the torus T^n with the flat square metric;
(ii) the sphere S^n with the standard metric;
(iii) the projective space $\mathbb{R}P^n$ with the standard metric.
Check in these examples that any point in the cut locus is either conjugate to p or joined to p by two geodesics with the same length but different images. (**Remark:** This is a general property of the cut locus—see [dC93] or [GHL04] for a proof).

3.5 Hopf-Rinow Theorem

Exercise 5.8

(1) Prove Proposition 5.4.

(2) Consider $\mathbb{R}^2 \setminus \{(x, 0) \mid -3 \leq x \leq 3\}$ with the Euclidean metric. Determine $B_7(0, 4)$.

(3) (a) Prove that a connected Riemannian manifold is complete if and only if the compact sets are the closed bounded sets.
(b) Give an example of a connected Riemannian manifold containing a non-compact closed bounded set.

(4) A Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is said to be **homogeneous** if, given any two points $p, q \in M$, there exists an isometry $f : M \to M$ such that $f(p) = q$. Show that:
(a) any homogeneous Riemannian manifold is complete;
(b) if G is a Lie group admitting a bi-invariant metric [cf. Exercise 4.8(3)] then the exponential map $\exp : \mathfrak{g} \to G$ is surjective;
(c) $SL(2, \mathbb{R})$ does not admit a bi-invariant metric.

(5) Let (M, g) be a complete Riemannian manifold. Show that:
(a) (*Ambrose theorem*) if (N, h) is a Riemannian manifold and $f : M \to N$ is a local isometry then f is a covering map;
(b) there exist surjective local isometries which are not covering maps;
(c) (*Cartan–Hadamard theorem*) if (M, g) has nonpositive curvature [cf. Exercise 4.8(6)] then for each point $p \in M$ the exponential map $\exp_p : T_pM \to M$ is a covering map. (**Remark:** In particular, if M is simply connected then M must be diffeomorphic to $\mathbb{R}^n$).