

**Exercise 1.8**

- (1) Which of the following sets (with the subspace topology) are topological manifolds?
- (a)  $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ ;
  - (b)  $S^2 \setminus \{p\} \ (p \in S^2)$ ;
  - (c)  $S^2 \setminus \{p, q\} \ (p, q \in S^2, p \neq q)$ ;
  - (d)  $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ ;
  - (e)  $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$ ;
- (2) Which of the manifolds above are homeomorphic?
- (3) Show that the Klein bottle  $K^2$  can be obtained by gluing two Möbius bands together through a homeomorphism of the boundary.
- (4) Show that:
- (a)  $M \# S^2 = M$  for any 2-dimensional topological manifold  $M$ ;
  - (b)  $\mathbb{R}P^2 \# \mathbb{R}P^2 = K^2$ ;
  - (c)  $\mathbb{R}P^2 \# T^2 = \mathbb{R}P^2 \# K^2$ .
- (5) A **triangulation** of a 2-dimensional topological manifold  $M$  is a decomposition of  $M$  in a finite number of triangles (i.e. subsets homeomorphic to triangles in  $\mathbb{R}^2$ ) such that the intersection of any two triangles is either a common edge, a common vertex or empty (it is possible to prove that such a triangulation always exists). The **Euler characteristic** of  $M$  is

$$\chi(M) := V - E + F,$$

where  $V$ ,  $E$  and  $F$  are the number of vertices, edges and faces of a given triangulation (it can be shown that this is well defined, i.e. does not depend on the choice of triangulation). Show that:

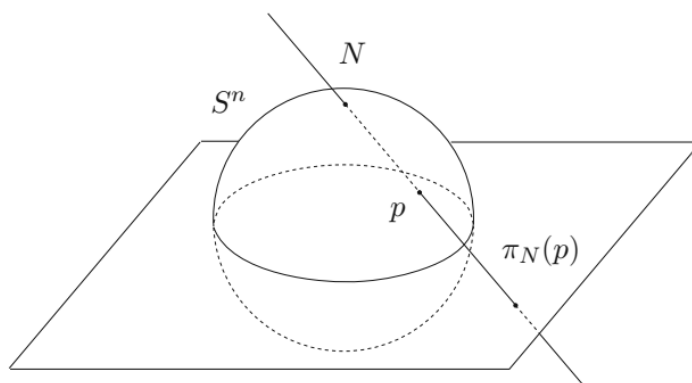
- (a) adding a vertex to a triangulation does not change  $\chi(M)$ ;
- (b)  $\chi(S^2) = 2$ ;
- (c)  $\chi(T^2) = 0$ ;
- (d)  $\chi(K^2) = 0$ ;
- (e)  $\chi(\mathbb{R}P^2) = 1$ ;
- (f)  $\chi(M \# N) = \chi(M) + \chi(N) - 2$ .

## 1.2 Differentiable Manifolds

### Exercise 2.5

- (1) Show that two atlases  $\mathcal{A}_1$  and  $\mathcal{A}_2$  for a smooth manifold are equivalent if and only if  $\mathcal{A}_1 \cup \mathcal{A}_2$  is an atlas.
- (2) Let  $M$  be a differentiable manifold. Show that a set  $V \subset M$  is open if and only if  $\varphi_\alpha^{-1}(V)$  is an open subset of  $\mathbb{R}^n$  for every parameterization  $(U_\alpha, \varphi_\alpha)$  of a  $C^\infty$  atlas.
- (3) Show that the two atlases on  $\mathbb{R}^n$  from Example 2.3(1) are equivalent.
- (4) Consider the two atlases on  $\mathbb{R}$  from Example 2.3(2),  $\{(\mathbb{R}, \varphi_1)\}$  and  $\{(\mathbb{R}, \varphi_2)\}$ , where  $\varphi_1(x) = x$  and  $\varphi_2(x) = x^3$ . Show that  $\varphi_2^{-1} \circ \varphi_1$  is not differentiable at the origin. Conclude that the two atlases are not equivalent.
- (5) Recall from elementary vector calculus that a **surface**  $S \subset \mathbb{R}^3$  is a set such that, for each  $p \in S$ , there is a neighborhood  $V_p$  of  $p$  in  $\mathbb{R}^3$  and a  $C^\infty$  map  $f_p : U_p \rightarrow \mathbb{R}$  (where  $U_p$  is an open subset of  $\mathbb{R}^2$ ) such that  $S \cap V_p$  is the graph of  $z = f_p(x, y)$ , or  $x = f_p(y, z)$ , or  $y = f_p(x, z)$ . Show that  $S$  is a smooth manifold of dimension 2.
- (6) (Product manifold) Let  $\{(U_\alpha, \varphi_\alpha)\}, \{(V_\beta, \psi_\beta)\}$  be two atlases for two smooth manifolds  $M$  and  $N$ . Show that the family  $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$  is an atlas for the product  $M \times N$ . With the differentiable structure generated by this atlas,  $M \times N$  is called the **product manifold** of  $M$  and  $N$ .
- (7) (Stereographic projection) Consider the  $n$ -sphere  $S^n$  with the subspace topology and let  $N = (0, \dots, 0, 1)$  and  $S = (0, \dots, 0, -1)$  be the north and south poles. The **stereographic projection** from  $N$  is the map  $\pi_N : S^n \setminus \{N\} \rightarrow \mathbb{R}^n$  which takes a point  $p \in S^n \setminus \{N\}$  to the intersection point of the line through  $N$  and  $p$  with the hyperplane  $x^{n+1} = 0$  (cf. Fig. 1.10). Similarly, the stereographic projection from  $S$  is the map  $\pi_S : S^n \setminus \{S\} \rightarrow \mathbb{R}^n$  which takes a point  $p$  on  $S^n \setminus \{S\}$  to the intersection point of the line through  $S$  and  $p$  with the same hyperplane. Check that  $\{(\mathbb{R}^n, \pi_N^{-1}), (\mathbb{R}^n, \pi_S^{-1})\}$  is an atlas for  $S^n$ . Show that this atlas is equivalent to the atlas on Example 2.3(5). The maximal atlas obtained from these is called the **standard differentiable structure** on  $S^n$ .

- (8) (*Real projective space*) The **real projective space**  $\mathbb{R}P^n$  is the set of lines through the origin in  $\mathbb{R}^{n+1}$ . This space can be defined as the quotient space of  $S^n$  by the equivalence relation  $x \sim -x$  that identifies a point to its antipodal point.
- (a) Show that the quotient space  $\mathbb{R}P^n = S^n / \sim$  with the quotient topology is a Hausdorff space and admits a countable basis of open sets. (**Hint:** Use Proposition 10.2).
- (b) Considering the atlas on  $S^n$  defined in Example 2.3(5) and the canonical projection  $\pi : S^n \rightarrow \mathbb{R}P^n$  given by  $\pi(x) = [x]$ , define an atlas for  $\mathbb{R}P^n$ .
- (9) We can define an atlas on  $\mathbb{R}P^n$  in a different way by identifying it with the quotient space of  $\mathbb{R}^{n+1} \setminus \{0\}$  by the equivalence relation  $x \sim \lambda x$ , with  $\lambda \in \mathbb{R} \setminus \{0\}$ . For that, consider the sets  $V_i = \{[x^1, \dots, x^{n+1}] \mid x^i \neq 0\}$  (corresponding to the



set of lines through the origin in  $\mathbb{R}^{n+1}$  that are not contained on the hyperplane  $x^i = 0$ ) and the maps  $\varphi_i : \mathbb{R}^n \rightarrow V_i$  defined by

$$\varphi_i(x^1, \dots, x^n) = [x^1, \dots, x^{i-1}, 1, x^i, \dots, x^n].$$

Show that:

- (a) the family  $\{(\mathbb{R}^n, \varphi_i)\}$  is an atlas for  $\mathbb{R}P^n$ ;
- (b) this atlas defines the same differentiable structure as the atlas on Exercise 2.5(8).
- (10) (*A non-Hausdorff manifold*) Let  $M$  be the disjoint union of  $\mathbb{R}$  with a point  $p$  and consider the maps  $f_i : \mathbb{R} \rightarrow M$  ( $i = 1, 2$ ) defined by  $f_i(x) = x$  if  $x \in \mathbb{R} \setminus \{0\}$ ,  $f_1(0) = 0$  and  $f_2(0) = p$ . Show that:
- (a) the maps  $f_i^{-1} \circ f_j$  are differentiable on their domains;
- (b) if we consider an atlas formed by  $\{(\mathbb{R}, f_1), (\mathbb{R}, f_2)\}$ , the corresponding topology will not satisfy the Hausdorff axiom.

### 1.3 Differentiable maps

#### Exercise 3.2

- (1) Prove that Definition 3.1 does not depend on the choice of parameterizations.
- (2) Show that a differentiable map  $f : M \rightarrow N$  between two smooth manifolds is continuous.
- (3) Show that if  $f : M_1 \rightarrow M_2$  and  $g : M_2 \rightarrow M_3$  are differentiable maps between smooth manifolds  $M_1, M_2$  and  $M_3$ , then  $g \circ f : M_1 \rightarrow M_3$  is also differentiable.
- (4) Show that the **antipodal map**  $f : S^n \rightarrow S^n$ , defined by  $f(x) = -x$ , is differentiable.
- (5) Using the stereographic projection from the north pole  $\pi_N : S^2 \setminus \{N\} \rightarrow \mathbb{R}^2$  and identifying  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$ , we can identify  $S^2$  with  $\mathbb{C} \cup \{\infty\}$ , where  $\infty$  is the so-called **point at infinity**. A **Möbius transformation** is a map  $f : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$  of the form

$$f(z) = \frac{az + b}{cz + d},$$

where  $a, b, c, d \in \mathbb{C}$  satisfy  $ad - bc \neq 0$  and  $\infty$  satisfies

$$\frac{\alpha}{\infty} = 0, \quad \frac{\alpha}{0} = \infty$$

for any  $\alpha \in \mathbb{C} \setminus \{0\}$ . Show that any Möbius transformation  $f$ , seen as a map  $f : S^2 \rightarrow S^2$ , is a diffeomorphism. (**Hint:** Start by showing that any Möbius transformation is a composition of transformations of the form  $g(z) = \frac{1}{z}$  and  $h(z) = az + b$ ).

- (6) Consider again the two atlases on  $\mathbb{R}$  from Example 2.3(2) and Exercise 2.5(4),  $\{(\mathbb{R}, \varphi_1)\}$  and  $\{(\mathbb{R}, \varphi_2)\}$ , where  $\varphi_1(x) = x$  and  $\varphi_2(x) = x^3$ . Show that:
  - (a) the identity map  $i : (\mathbb{R}, \varphi_1) \rightarrow (\mathbb{R}, \varphi_2)$  is not a diffeomorphism;
  - (b) the map  $f : (\mathbb{R}, \varphi_1) \rightarrow (\mathbb{R}, \varphi_2)$  defined by  $f(x) = x^3$  is a diffeomorphism (implying that although these two atlases define different differentiable structures, they are diffeomorphic).

### 1.4 Tangent Space

#### Exercise 4.9

- (1) Show that the operators  $\left(\frac{\partial}{\partial x^i}\right)_p$  are linearly independent.

- (2) Let  $M$  be a smooth manifold,  $p$  a point in  $M$  and  $v$  a vector tangent to  $M$  at  $p$ . Show that if  $v$  can be written as  $v = \sum_{i=1}^n a^i \left( \frac{\partial}{\partial x^i} \right)_p$  and  $v = \sum_{i=1}^n b^i \left( \frac{\partial}{\partial y^i} \right)_p$  for two basis associated to different parameterizations around  $p$ , then

$$b^j = \sum_{i=1}^n \frac{\partial y^j}{\partial x^i} a^i.$$

- (3) Let  $M$  be an  $n$ -dimensional differentiable manifold and  $p \in M$ . Show that the following sets can be canonically identified with  $T_p M$  (and therefore constitute alternative definitions of the tangent space):

- (a)  $\mathcal{C}_p / \sim$ , where  $\mathcal{C}_p$  is the set of differentiable curves  $c : I \subset \mathbb{R} \rightarrow M$  such that  $c(0) = p$  and  $\sim$  is the equivalence relation defined by

$$c_1 \sim c_2 \Leftrightarrow \frac{d}{dt} \left( \varphi^{-1} \circ c_1 \right) (0) = \frac{d}{dt} \left( \varphi^{-1} \circ c_2 \right) (0)$$

for some parameterization  $\varphi : U \subset \mathbb{R}^n \rightarrow M$  of a neighborhood of  $p$ .

- (b)  $\{(\alpha, v_\alpha) \mid p \in \varphi_\alpha(U_\alpha) \text{ and } v_\alpha \in \mathbb{R}^n\} / \sim$ , where  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$  is the differentiable structure and  $\sim$  is the equivalence relation defined by

$$(\alpha, v_\alpha) \sim (\beta, v_\beta) \Leftrightarrow v_\beta = d \left( \varphi_\beta^{-1} \circ \varphi_\alpha \right)_{\varphi_\alpha^{-1}(p)} (v_\alpha).$$

- (4) (*Chain rule*) Let  $f : M \rightarrow N$  and  $g : N \rightarrow P$  be two differentiable maps. Then  $g \circ f : M \rightarrow P$  is also differentiable [cf. Exercise 3.2(3)]. Show that for  $p \in M$ ,

$$(d(g \circ f))_p = (dg)_{f(p)} \circ (df)_p.$$

- (5) Let  $\phi : (0, +\infty) \times (0, \pi) \times (0, 2\pi) \rightarrow \mathbb{R}^3$  be the parameterization of  $U = \mathbb{R}^3 \setminus \{(x, 0, z) \mid x \geq 0 \text{ and } z \in \mathbb{R}\}$  by spherical coordinates,

$$\phi(r, \theta, \varphi) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta).$$

Determine the Cartesian components of  $\frac{\partial}{\partial r}$ ,  $\frac{\partial}{\partial \theta}$  and  $\frac{\partial}{\partial \varphi}$  at each point of  $U$ .

- (6) Compute the derivative  $(df)_N$  of the antipodal map  $f : S^n \rightarrow S^n$  at the north pole  $N$ .

- (7) Let  $W$  be a coordinate neighborhood on  $M$ , let  $x : W \rightarrow \mathbb{R}^n$  be a coordinate chart and consider a smooth function  $f : M \rightarrow \mathbb{R}$ . Show that for  $p \in W$ , the derivative  $(df)_p$  is given by

$$(df)_p = \frac{\partial \hat{f}}{\partial x^1}(x(p)) (dx^1)_p + \cdots + \frac{\partial \hat{f}}{\partial x^n}(x(p)) (dx^n)_p,$$

where  $\hat{f} := f \circ x^{-1}$ .

- (8) (Tangent bundle) Let  $\{(U_\alpha, \varphi_\alpha)\}$  be a differentiable structure on  $M$  and consider the maps

$$\begin{aligned} \Phi_\alpha : U_\alpha \times \mathbb{R}^n &\rightarrow TM \\ (x, v) &\mapsto (d\varphi_\alpha)_x(v) \in T_{\varphi_\alpha(x)}M. \end{aligned}$$

Show that the family  $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$  defines a differentiable structure for  $TM$ . Conclude that, with this differentiable structure,  $TM$  is a smooth manifold of dimension  $2 \times \dim M$ .

- (9) Let  $f : M \rightarrow N$  be a differentiable map between smooth manifolds. Show that:

- (a)  $df : TM \rightarrow TN$  is also differentiable;
- (b) if  $f : M \rightarrow M$  is the identity map then  $df : TM \rightarrow TM$  is also the identity;
- (c) if  $f$  is a diffeomorphism then  $df : TM \rightarrow TN$  is also a diffeomorphism and  $(df)^{-1} = df^{-1}$ .

- (10) Let  $M_1, M_2$  be two differentiable manifolds and

$$\begin{aligned} \pi_1 : M_1 \times M_2 &\rightarrow M_1 \\ \pi_2 : M_1 \times M_2 &\rightarrow M_2 \end{aligned}$$

the corresponding canonical projections.

- (a) Show that  $d\pi_1 \times d\pi_2$  is a diffeomorphism between the tangent bundle  $T(M_1 \times M_2)$  and the product manifold  $TM_1 \times TM_2$ .
- (b) Show that if  $N$  is a smooth manifold and  $f_i : N \rightarrow M_i$  ( $i = 1, 2$ ) are differentiable maps, then  $d(f_1 \times f_2) = df_1 \times df_2$ .

## 1.5 Immersions and Embeddings

### Exercise 5.9

- (1) Show that any parameterization  $\varphi : U \subset \mathbb{R}^m \rightarrow M$  is an embedding of  $U$  into  $M$ .

- (2) Show that, locally, any immersion is an embedding, i.e. if  $f : M \rightarrow N$  is an immersion and  $p \in M$ , then there is an open set  $W \subset M$  containing  $p$  such that  $f|_W$  is an embedding.
- (3) Let  $N$  be a manifold. Show that  $M \subset N$  is a submanifold of  $N$  of dimension  $m$  if and only if, for each  $p \in M$ , there is a coordinate system  $x : W \rightarrow \mathbb{R}^n$  around  $p$  on  $N$ , for which  $M \cap W$  is defined by the equations  $x^{m+1} = \dots = x^n = 0$ .
- (4) Consider the sphere

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \mid (x^1)^2 + \dots + (x^{n+1})^2 = 1 \right\}.$$

Show that  $S^n$  is an  $n$ -dimensional submanifold of  $\mathbb{R}^{n+1}$  and that

$$T_x S^n = \left\{ v \in \mathbb{R}^{n+1} \mid \langle x, v \rangle = 0 \right\},$$

where  $\langle \cdot, \cdot \rangle$  is the usual inner product on  $\mathbb{R}^n$ .

- (5) Let  $f : M \rightarrow N$  be a differentiable map between smooth manifolds and consider submanifolds  $V \subset M$  and  $W \subset N$ . Show that if  $f(V) \subset W$  then  $f : V \rightarrow W$  is also a differentiable map.
- (6) Let  $f : M \rightarrow N$  be an injective immersion. Show that if  $M$  is compact then  $f(M)$  is a submanifold of  $N$ .

## 1.6 Vector Fields

### Exercise 6.11

- (1) Let  $X : M \rightarrow TM$  be a differentiable vector field on  $M$  and, for a smooth function  $f : M \rightarrow \mathbb{R}$ , consider its directional derivative along  $X$  defined by

$$\begin{aligned} X \cdot f : M &\rightarrow \mathbb{R} \\ p &\mapsto X_p \cdot f. \end{aligned}$$

Show that:

- (a)  $(X \cdot f)(p) = (df)_p X_p$ ;
- (b) the vector field  $X$  is smooth if and only if  $X \cdot f$  is a differentiable function for any smooth function  $f : M \rightarrow \mathbb{R}$ ;

- (c) the directional derivative satisfies the following properties: for  $f, g \in C^\infty(M)$  and  $\alpha \in \mathbb{R}$ ,
- (i)  $X \cdot (f + g) = X \cdot f + X \cdot g$ ;
  - (ii)  $X \cdot (\alpha f) = \alpha(X \cdot f)$ ;
  - (iii)  $X \cdot (fg) = fX \cdot g + gX \cdot f$ .

(2) Prove Proposition 6.3.

(3) Show that  $(\mathbb{R}^3, \times)$  is a Lie algebra, where  $\times$  is the cross product on  $\mathbb{R}^3$ .

(4) Compute the flows of the vector fields  $X, Y, Z \in \mathfrak{X}(\mathbb{R}^2)$  defined by

$$X_{(x,y)} = \frac{\partial}{\partial x}; \quad Y_{(x,y)} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}; \quad Z_{(x,y)} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$



[RG1101vectorfield01 p.4]

(5) Let  $X_1, X_2, X_3 \in \mathfrak{X}(\mathbb{R}^3)$  be the vector fields defined by

$$X_1 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad X_2 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}, \quad X_3 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},$$

where  $(x, y, z)$  are the usual Cartesian coordinates.

- (a) Compute the Lie brackets  $[X_i, X_j]$  for  $i, j = 1, 2, 3$ .
- (b) Show that  $\text{span}\{X_1, X_2, X_3\}$  is a Lie subalgebra of  $\mathfrak{X}(\mathbb{R}^3)$ , isomorphic to  $(\mathbb{R}^3, \times)$ .
- (c) Compute the flows  $\psi_{1,t}, \psi_{2,t}, \psi_{3,t}$  of  $X_1, X_2, X_3$ .
- (d) Show that  $\psi_{i, \frac{\pi}{2}} \circ \psi_{j, \frac{\pi}{2}} \neq \psi_{j, \frac{\pi}{2}} \circ \psi_{i, \frac{\pi}{2}}$  for  $i \neq j$ .



[RG1101vectorfield01 p.6]

(6) Give an example of a non-complete vector field.



[RG1101CompleteVectorFields]

(7) Let  $N$  be a differentiable manifold,  $M \subset N$  a submanifold and  $X, Y \in \mathfrak{X}(N)$  vector fields tangent to  $M$ , i.e. such that  $X_p, Y_p \in T_p M$  for all  $p \in M$ . Show that  $[X, Y]$  is also tangent to  $M$ , and that its restriction to  $M$  coincides with the Lie bracket of the restrictions of  $X$  and  $Y$  to  $M$ .

(8) Let  $f : M \rightarrow N$  be a smooth map between manifolds. Two vector fields  $X \in \mathfrak{X}(M)$  and  $Y \in \mathfrak{X}(N)$  are said to be  **$f$ -related** (and we write  $Y = f_* X$ ) if, for each  $q \in N$  and  $p \in f^{-1}(q) \subset M$ , we have  $(df)_p X_p = Y_q$ . Show that:



- (a) given  $f$  and  $X$  it is possible that no vector field  $Y$  is  $f$ -related to  $X$ ;
- (b) the vector field  $X$  is  $f$ -related to  $Y$  if and only if, for any differentiable function  $g$  defined on some open subset  $W$  of  $N$ ,  $(Y \cdot g) \circ f = X \cdot (g \circ f)$  on the inverse image  $f^{-1}(W)$  of the domain of  $g$ ;
- (c) for differentiable maps  $f : M \rightarrow N$  and  $g : N \rightarrow P$  between smooth manifolds and vector fields  $X \in \mathfrak{X}(M)$ ,  $Y \in \mathfrak{X}(N)$  and  $Z \in \mathfrak{X}(P)$ , if  $X$  is  $f$ -related to  $Y$  and  $Y$  is  $g$ -related to  $Z$ , then  $X$  is  $(g \circ f)$ -related to  $Z$ .
- (9) Let  $f : M \rightarrow N$  be a diffeomorphism between smooth manifolds. Show that  $f_*[X, Y] = [f_*X, f_*Y]$  for every  $X, Y \in \mathfrak{X}(M)$ . Therefore,  $f_*$  induces a Lie algebra isomorphism between  $\mathfrak{X}(M)$  and  $\mathfrak{X}(N)$ .
- (10) Let  $f : M \rightarrow N$  be a differentiable map between smooth manifolds and consider two vector fields  $X \in \mathfrak{X}(M)$  and  $Y \in \mathfrak{X}(N)$ . Show that:
- (a) if the vector field  $Y$  is  $f$ -related to  $X$  then any integral curve of  $X$  is mapped by  $f$  into an integral curve of  $Y$ ;
- (b) the vector field  $Y$  is  $f$ -related to  $X$  if and only if the local flows  $F_X$  and  $F_Y$  satisfy  $f(F_X(p, t)) = F_Y(f(p), t)$  for all  $(t, p)$  for which both sides are defined.
- (11) (*Lie derivative of a function*) Given a vector field  $X \in \mathfrak{X}(M)$ , we define the **Lie derivative** of a smooth function  $f : M \rightarrow \mathbb{R}$  in the direction of  $X$  as

$$L_X f(p) := \frac{d}{dt} ((f \circ \psi_t)(p)) \Big|_{t=0},$$

where  $\psi_t = F(\cdot, t)$ , for  $F$  the local flow of  $X$  at  $p$ . Show that  $L_X f = X \cdot f$ , meaning that the Lie derivative of  $f$  in the direction of  $X$  is just the directional derivative of  $f$  along  $X$ .

- (12) (*Lie derivative of a vector field*) For two vector fields  $X, Y \in \mathfrak{X}(M)$  we define the **Lie derivative** of  $Y$  in the direction of  $X$  as

$$L_X Y := \frac{d}{dt} ((\psi_{-t})_* Y) \Big|_{t=0},$$

where  $\{\psi_t\}_{t \in I}$  is the local flow of  $X$ . Show that:

- (a)  $L_X Y = [X, Y]$ ;
- (b)  $L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z]$ , for  $X, Y, Z \in \mathfrak{X}(M)$ ;
- (c)  $L_X \circ L_Y - L_Y \circ L_X = L_{[X, Y]}$ .



- (13) Let  $X, Y \in \mathfrak{X}(M)$  be two complete vector fields with flows  $\psi, \phi$ . Show that:
- (a) given a diffeomorphism  $f : M \rightarrow M$ , we have  $f_*X = X$  if and only if  $f \circ \psi_t = \psi_t \circ f$  for all  $t \in \mathbb{R}$ ;
  - (b)  $\psi_t \circ \phi_s = \phi_s \circ \psi_t$  for all  $s, t \in \mathbb{R}$  if and only if  $[X, Y] = 0$ .

## 1.7 Lie Groups

### Exercise 7.17

- (1) (a) Given two Lie groups  $G_1, G_2$ , show that  $G_1 \times G_2$  (the direct product of the two groups) is a Lie group with the standard differentiable structure on the product.
- (b) The circle  $S^1$  can be identified with the set of complex numbers of absolute value 1. Show that  $S^1$  is a Lie group and conclude that the  $n$ -torus  $T^n \cong S^1 \times \dots \times S^1$  is also a Lie group.
- (2) (a) Show that  $(\mathbb{R}^n, +)$  is a Lie group, determine its Lie algebra and write an expression for the exponential map.
- (b) Prove that, if  $G$  is an abelian Lie group, then  $[V, W] = 0$  for all  $V, W \in \mathfrak{g}$ .

- (3) We can identify each point in

$$H = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

with the invertible affine map  $h : \mathbb{R} \rightarrow \mathbb{R}$  given by  $h(t) = yt + x$ . The set of all such maps is a group under composition; consequently, our identification induces a group structure on  $H$ .

- (a) Show that the induced group operation is given by

$$(x, y) \cdot (z, w) = (yz + x, yw),$$

and that  $H$ , with this group operation, is a Lie group.

- (b) Show that the derivative of the left translation map  $L_{(x,y)} : H \rightarrow H$  at a point  $(z, w) \in H$  is represented in the above coordinates by the matrix

$$(dL_{(x,y)})_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

Conclude that the left-invariant vector field  $X^V \in \mathfrak{X}(H)$  determined by the vector

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \in \mathfrak{h} \equiv T_{(0,1)}H \quad (\xi, \eta \in \mathbb{R})$$

is given by

$$X_{(x,y)}^V = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y}.$$

- (c) Given  $V, W \in \mathfrak{h}$ , compute  $[V, W]$ .  
 (d) Determine the flow of the vector field  $X^V$ , and give an expression for the exponential map  $\exp : \mathfrak{h} \rightarrow H$ .  
 (e) Confirm your results by first showing that  $H$  is the subgroup of  $GL(2)$  formed by the matrices

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

with  $y > 0$ .



[RG4701hyperbolicplane p.3]

- (4) Consider the group

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\},$$

which we already know to be a 3-manifold. Making

$$a = p + q, d = p - q, b = r + s, c = r - s,$$

show that  $SL(2)$  is diffeomorphic to  $S^1 \times \mathbb{R}^2$ .

- (5) Give examples of matrices  $A, B \in \mathfrak{gl}(2)$  such that  $e^{A+B} \neq e^A e^B$ .

(6) For  $A \in \mathfrak{gl}(n)$ , consider the differentiable map

$$\begin{aligned} h : \mathbb{R} &\rightarrow \mathbb{R} \setminus \{0\} \\ t &\mapsto \det e^{At} \end{aligned}$$

and show that:

- (a) this map is a group homomorphism between  $(\mathbb{R}, +)$  and  $(\mathbb{R} \setminus \{0\}, \cdot)$ ;
- (b)  $h'(0) = \text{tr} A$ ;
- (c)  $\det(e^A) = e^{\text{tr} A}$ .

(7) (a) If  $A \in \mathfrak{sl}(2)$ , show that there is a  $\lambda \in \mathbb{R} \cup i\mathbb{R}$  such that

$$e^A = \cosh \lambda \, I + \frac{\sinh \lambda}{\lambda} A.$$

(b) Show that  $\exp : \mathfrak{sl}(2) \rightarrow SL(2)$  is not surjective.

(8) Consider the vector field  $X \in \mathfrak{X}(\mathbb{R}^2)$  defined by

$$X = \sqrt{x^2 + y^2} \frac{\partial}{\partial x}.$$

- (a) Show that the flow of  $X$  defines a free action of  $\mathbb{R}$  on  $M = \mathbb{R}^2 \setminus \{0\}$ .
- (b) Describe the topological quotient space  $M/\mathbb{R}$ . Is the action above proper?

(9) Let  $M = S^2 \times S^2$  and consider the diagonal  $S^1$ -action on  $M$  given by

$$e^{i\theta} \cdot (u, v) = (e^{i\theta} \cdot u, e^{2i\theta} \cdot v),$$

where, for  $u \in S^2 \subset \mathbb{R}^3$  and  $e^{i\beta} \in S^1$ ,  $e^{i\beta} \cdot u$  denotes the rotation of  $u$  by an angle  $\beta$  around the  $z$ -axis.

- (a) Determine the fixed points for this action.
- (b) What are the possible nontrivial stabilizers?

(10) Let  $G$  be a Lie group and  $H$  a closed Lie subgroup, i.e. a subgroup of  $G$  which is also a closed submanifold of  $G$ . Show that the action of  $H$  in  $G$  defined by  $A(h, g) = h \cdot g$  is free and proper.

- (11) (*Grassmannian*) Consider the set  $H \subset GL(n)$  of invertible matrices of the form

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

where  $A \in GL(k)$ ,  $B \in GL(n-k)$  and  $C \in \mathcal{M}_{(n-k) \times k}$ .

- (a) Show that  $H$  is a closed Lie subgroup of  $GL(n)$ . Therefore  $H$  acts freely and properly on  $GL(n)$  [cf. Exercise 7.17(10)].
- (b) Show that the quotient manifold

$$Gr(n, k) := GL(n)/H$$

can be identified with the set of  $k$ -dimensional subspaces of  $\mathbb{R}^n$  (in particular  $Gr(n, 1)$  is just the projective space  $\mathbb{R}P^{n-1}$ ).

- (c) The manifold  $Gr(n, k)$  is called the **Grassmannian** of  $k$ -planes in  $\mathbb{R}^n$ . What is its dimension?

- (12) Let  $G$  and  $H$  be Lie groups and  $F : G \rightarrow H$  a Lie group homomorphism. Show that:

- (a)  $(dF)_e : \mathfrak{g} \rightarrow \mathfrak{h}$  is a Lie algebra homomorphism;
- (b) if  $(dF)_e$  is an isomorphism then  $F$  is a local diffeomorphism;
- (c) if  $F$  is a surjective local diffeomorphism then  $F$  is a covering map.

- (13) (a) Show that  $\mathbb{R} \cdot SU(2)$  is a four-dimensional real linear subspace of  $\mathcal{M}_{2 \times 2}(\mathbb{C})$ , closed under matrix multiplication, with basis

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying  $i^2 = j^2 = k^2 = ijk = -1$ . Therefore this space can be identified with the **quaternions** (cf. Sect. 1.10.1). Show that  $SU(2)$  can be identified with the quaternions of Euclidean norm equal to 1, and is therefore diffeomorphic to  $S^3$ .

- (b) Show that if  $n \in \mathbb{R}^3$  is a unit vector, which we identify with a quaternion with zero real part, then

$$\exp\left(\frac{n\theta}{2}\right) = 1 \cos\left(\frac{\theta}{2}\right) + n \sin\left(\frac{\theta}{2}\right)$$

is also a unit quaternion.

- (c) Again identifying  $\mathbb{R}^3$  with quaternions with zero real part, show that the map

$$\begin{aligned}\mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ v &\mapsto \exp\left(\frac{n\theta}{2}\right) \cdot v \cdot \exp\left(-\frac{n\theta}{2}\right)\end{aligned}$$

is a rotation by an angle  $\theta$  about the axis defined by  $n$ .

- (d) Show that there exists a surjective homomorphism  $F : SU(2) \rightarrow SO(3)$ , and use this to conclude that  $SU(2)$  is the universal covering of  $SO(3)$ .
- (e) What is the fundamental group of  $SO(3)$ ?

## 1.8 Orientability

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## 1.9 Manifolds with Boundary

### Exercise 9.5

- (1) Show with an example that the product of two manifolds with boundary is not always a manifold with boundary.
- (2) Let  $M$  be a manifold without boundary and  $N$  a manifold with boundary. Show that the product  $M \times N$  is a manifold with boundary. What is  $\partial(M \times N)$ ?
- (3) Show that a diffeomorphism between two manifolds with boundary  $M$  and  $N$  maps the boundary  $\partial M$  diffeomorphically onto  $\partial N$ .

後面有 **Notes** 是各節的補充說明。