An introduction to Riemannian Geometry Jose Natari

Exercise 1.8

(1) Which of the following sets (with the subspace topology) are topological manifolds?

(a) $D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$;
(b) $S^2 \setminus \{p\}$ $(p \in S^2)$;
(c) $S^2 \setminus \{p, q\}$ $(p, q \in S^2, p \neq q)$;
(d) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$;
(e) $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$;

(2) Which of the manifolds above are homeomorphic?

(3) Show that the Klein bottle K^2 can be obtained by gluing two Möbius bands together through a homeomorphism of the boundary.

(4) Show that:

(a) $M \# S^2 = M$ for any 2-dimensional topological manifold M;
(b) $\mathbb{R}P^2 \# \mathbb{R}P^2 = K^2$;
(c) $\mathbb{R}P^2 \# T^2 = \mathbb{R}P^2 \# K^2$.

(5) A **triangulation** of a 2-dimensional topological manifold M is a decomposition of M in a finite number of triangles (i.e. subsets homeomorphic to triangles in $\mathbb{R}^2$) such that the intersection of any two triangles is either a common edge, a common vertex or empty (it is possible to prove that such a triangulation always exists). The **Euler characteristic** of M is

$$\chi(M) := V - E + F,$$

where V, E and F are the number of vertices, edges and faces of a given triangulation (it can be shown that this is well defined, i.e. does not depend on the choice of triangulation). Show that:

(a) adding a vertex to a triangulation does not change $\chi(M)$;
(b) $\chi\left(S^2\right) = 2$;
(c) $\chi\left(T^2\right) = 0$;
(d) $\chi\left(K^2\right) = 0$;
(e) $\chi\left(\mathbb{R}P^2\right) = 1$;
(f) $\chi(M \# N) = \chi(M) + \chi(N) - 2$.

1.2 Differentiable Manifolds

Exercise 2.5

(1) Show that two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ for a smooth manifold are equivalent if and only if $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas.

(2) Let M be a differentiable manifold. Show that a set $V \subset M$ is open if and only if $\varphi_\alpha^{-1}(V)$ is an open subset of $\mathbb{R}^n$ for every parameterization $(U_\alpha, \varphi_\alpha)$ of a C^∞ atlas.

(3) Show that the two atlases on $\mathbb{R}^n$ from Example 2.3(1) are equivalent.

(4) Consider the two atlases on $\mathbb{R}$ from Example 2.3(2), $\{(\mathbb{R}, \varphi_1)\}$ and $\{(\mathbb{R}, \varphi_2)\}$, where $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$. Show that $\varphi_2^{-1} \circ \varphi_1$ is not differentiable at the origin. Conclude that the two atlases are not equivalent.

(5) Recall from elementary vector calculus that a **surface** $S \subset \mathbb{R}^3$ is a set such that, for each $p \in S$, there is a neighborhood V_p of p in $\mathbb{R}^3$ and a C^∞ map $f_p : U_p \to \mathbb{R}$ (where U_p is an open subset of $\mathbb{R}^2$) such that $S \cap V_p$ is the graph of $z = f_p(x, y)$, or $x = f_p(y, z)$, or $y = f_p(x, z)$. Show that S is a smooth manifold of dimension 2.

(6) (Product manifold) Let $\{(U_\alpha, \varphi_\alpha)\}$, $\{(V_\beta, \psi_\beta)\}$ be two atlases for two smooth manifolds M and N. Show that the family $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$ is an atlas for the product $M \times N$. With the differentiable structure generated by this atlas, $M \times N$ is called the **product manifold** of M and N.

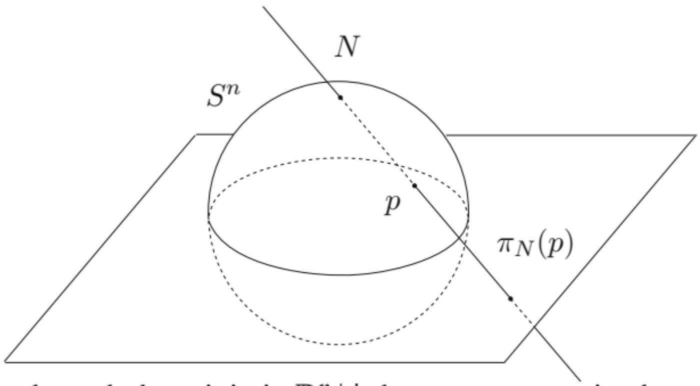
(7) (Stereographic projection) Consider the n-sphere S^n with the subspace topology and let $N = (0, \ldots, 0, 1)$ and $S = (0, \ldots, 0, -1)$ be the north and south poles. The **stereographic projection** from N is the map $\pi_N : S^n \backslash \{N\} \to \mathbb{R}^n$ which takes a point $p \in S^n \backslash \{N\}$ to the intersection point of the line through N and p with the hyperplane $x^{n+1} = 0$ (cf. Fig. 1.10). Similarly, the stereographic projection from S is the map $\pi_S : S^n \backslash \{S\} \to \mathbb{R}^n$ which takes a point p on $S^n \backslash \{S\}$ to the intersection point of the line through S and p with the same hyperplane. Check that $\{(\mathbb{R}^n, \pi_N^{-1}), (\mathbb{R}^n, \pi_S^{-1})\}$ is an atlas for S^n. Show that this atlas is equivalent to the atlas on Example 2.3(5). The maximal atlas obtained from these is called the **standard differentiable structure** on S^n.

(8) (*Real projective space*) The **real projective space** $\mathbb{R}P^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. This space can be defined as the quotient space of S^n by the equivalence relation $x \sim -x$ that identifies a point to its antipodal point.

(a) Show that the quotient space $\mathbb{R}P^n = S^n/\sim$ with the quotient topology is a Hausdorff space and admits a countable basis of open sets. (**Hint:** Use Proposition 10.2).

(b) Considering the atlas on S^n defined in Example 2.3(5) and the canonical projection $\pi : S^n \to \mathbb{R}P^n$ given by $\pi(x) = [x]$, define an atlas for $\mathbb{R}P^n$.

(9) We can define an atlas on $\mathbb{R}P^n$ in a different way by identifying it with the quotient space of $\mathbb{R}^{n+1}\backslash\{0\}$ by the equivalence relation $x \sim \lambda x$, with $\lambda \in \mathbb{R}\backslash\{0\}$. For that, consider the sets $V_i = \left\{\left[x^1, \dots, x^{n+1}\right] \mid x^i \neq 0\right\}$ (corresponding to the



set of lines through the origin in $\mathbb{R}^{n+1}$ that are not contained on the hyperplane $x^i = 0$) and the maps $\varphi_i : \mathbb{R}^n \to V_i$ defined by

$$\varphi_i\left(x^1, \dots, x^n\right) = \left[x^1, \dots, x^{i-1}, 1, x^i, \dots, x^n\right].$$

Show that:

(a) the family $\{(\mathbb{R}^n, \varphi_i)\}$ is an atlas for $\mathbb{R}P^n$;

(b) this atlas defines the same differentiable structure as the atlas on Exercise 2.5(8).

(10) (*A non-Hausdorff manifold*) Let M be the disjoint union of $\mathbb{R}$ with a point p and consider the maps $f_i : \mathbb{R} \to M$ $(i = 1, 2)$ defined by $f_i(x) = x$ if $x \in \mathbb{R}\backslash\{0\}$, $f_1(0) = 0$ and $f_2(0) = p$. Show that:

(a) the maps $f_i^{-1} \circ f_j$ are differentiable on their domains;

(b) if we consider an atlas formed by $\{(\mathbb{R}, f_1), (\mathbb{R}, f_2)\}$, the corresponding topology will not satisfy the Hausdorff axiom.

1.3 Differentiable maps

Exercise 3.2

(1) Prove that Definition 3.1 does not depend on the choice of parameterizations.

(2) Show that a differentiable map $f : M \to N$ between two smooth manifolds is continuous.

(3) Show that if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are differentiable maps between smooth manifolds M_1, M_2 and M_3, then $g \circ f : M_1 \to M_3$ is also differentiable.

(4) Show that the **antipodal map** $f : S^n \to S^n$, defined by $f(x) = -x$, is differentiable.

(5) Using the stereographic projection from the north pole $\pi_N : S^2 \setminus \{N\} \to \mathbb{R}^2$ and identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, we can identify S^2 with $\mathbb{C} \cup \{\infty\}$, where ∞ is the so-called **point at infinity**. A **Möbius transformation** is a map $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ of the form

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$ and ∞ satisfies

$$\frac{\alpha}{\infty} = 0, \qquad \frac{\alpha}{0} = \infty$$

for any $\alpha \in \mathbb{C} \setminus \{0\}$. Show that any Möbius transformation f, seen as a map $f : S^2 \to S^2$, is a diffeomorphism. (**Hint:** Start by showing that any Möbius transformation is a composition of transformations of the form $g(z) = \frac{1}{z}$ and $h(z) = az + b$).

(6) Consider again the two atlases on $\mathbb{R}$ from Example 2.3(2) and Exercise 2.5(4), $\{(\mathbb{R}, \varphi_1)\}$ and $\{(\mathbb{R}, \varphi_2)\}$, where $\varphi_1(x) = x$ and $\varphi_2(x) = x^3$. Show that:

(a) the identity map $i : (\mathbb{R}, \varphi_1) \to (\mathbb{R}, \varphi_2)$ is not a diffeomorphism;
(b) the map $f : (\mathbb{R}, \varphi_1) \to (\mathbb{R}, \varphi_2)$ defined by $f(x) = x^3$ is a diffeomorphism (implying that although these two atlases define different differentiable structures, they are diffeomorphic).

1.4 Tangent Space

Exercise 4.9

(1) Show that the operators $\left(\frac{\partial}{\partial x^i}\right)_p$ are linearly independent.

(2) Let M be a smooth manifold, p a point in M and v a vector tangent to M at p. Show that if v can be written as $v = \sum_{i=1}^{n} a^i \left(\frac{\partial}{\partial x^i}\right)_p$ and $v = \sum_{i=1}^{n} b^i \left(\frac{\partial}{\partial y^i}\right)_p$ for two basis associated to different parameterizations around p, then

$$b^j = \sum_{i=1}^{n} \frac{\partial y^j}{\partial x^i} a^i.$$

(3) Let M be an n-dimensional differentiable manifold and $p \in M$. Show that the following sets can be canonically identified with T_pM (and therefore constitute alternative definitions of the tangent space):

(a) $\mathcal{C}_p/\sim$, where $\mathcal{C}_p$ is the set of differentiable curves $c : I \subset \mathbb{R} \to M$ such that $c(0) = p$ and $\sim$ is the equivalence relation defined by

$$c_1 \sim c_2 \Leftrightarrow \frac{d}{dt}\left(\varphi^{-1} \circ c_1\right)(0) = \frac{d}{dt}\left(\varphi^{-1} \circ c_2\right)(0)$$

for some parameterization $\varphi : U \subset \mathbb{R}^n \to M$ of a neighborhood of p.

(b) $\{(\alpha, v_\alpha) \mid p \in \varphi_\alpha(U_\alpha) \text{ and } v_\alpha \in \mathbb{R}^n\}/\sim$, where $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is the differentiable structure and $\sim$ is the equivalence relation defined by

$$(\alpha, v_\alpha) \sim (\beta, v_\beta) \Leftrightarrow v_\beta = d\left(\varphi_\beta^{-1} \circ \varphi_\alpha\right)_{\varphi_\alpha^{-1}(p)}(v_\alpha).$$

(4) (*Chain rule*) Let $f : M \to N$ and $g : N \to P$ be two differentiable maps. Then $g \circ f : M \to P$ is also differentiable [cf. Exercise 3.2(3)]. Show that for $p \in M$,

$$(d(g \circ f))_p = (dg)_{f(p)} \circ (df)_p.$$

(5) Let $\phi : (0, +\infty) \times (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ be the parameterization of $U = \mathbb{R}^3 \setminus \{(x, 0, z) \mid x \geq 0 \text{ and } z \in \mathbb{R}\}$ by spherical coordinates,

$$\phi(r, \theta, \varphi) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r \cos\theta).$$

Determine the Cartesian components of $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial \varphi}$ at each point of U.

(6) Compute the derivative $(df)_N$ of the antipodal map $f : S^n \to S^n$ at the north pole N.

(7) Let W be a coordinate neighborhood on M, let $x : W \to \mathbb{R}^n$ be a coordinate chart and consider a smooth function $f : M \to \mathbb{R}$. Show that for $p \in W$, the derivative $(df)_p$ is given by

$$(df)_p = \frac{\partial \hat{f}}{\partial x^1}(x(p)) \left(dx^1\right)_p + \cdots + \frac{\partial \hat{f}}{\partial x^n}(x(p)) \left(dx^n\right)_p,$$

where $\hat{f} := f \circ x^{-1}$.

(8) (Tangent bundle) Let $\{(U_\alpha, \varphi_\alpha)\}$ be a differentiable structure on M and consider the maps

$$\begin{aligned} \Phi_\alpha : U_\alpha \times \mathbb{R}^n &\to TM \\ (x, v) &\mapsto (d\varphi_\alpha)_x(v) \in T_{\varphi_\alpha(x)}M. \end{aligned}$$

Show that the family $\{(U_\alpha \times \mathbb{R}^n, \Phi_\alpha)\}$ defines a differentiable structure for TM. Conclude that, with this differentiable structure, TM is a smooth manifold of dimension $2 \times \dim M$.

(9) Let $f : M \to N$ be a differentiable map between smooth manifolds. Show that:

(a) $df : TM \to TN$ is also differentiable;
(b) if $f : M \to M$ is the identity map then $df : TM \to TM$ is also the identity;
(c) if f is a diffeomorphism then $df : TM \to TN$ is also a diffeomorphism and $(df)^{-1} = df^{-1}$.

(10) Let M_1, M_2 be two differentiable manifolds and

$$\begin{aligned} \pi_1 &: M_1 \times M_2 \to M_1 \\ \pi_2 &: M_1 \times M_2 \to M_2 \end{aligned}$$

the corresponding canonical projections.

(a) Show that $d\pi_1 \times d\pi_2$ is a diffeomorphism between the tangent bundle $T(M_1 \times M_2)$ and the product manifold $TM_1 \times TM_2$.
(b) Show that if N is a smooth manifold and $f_i : N \to M_i$ $(i = 1, 2)$ are differentiable maps, then $d(f_1 \times f_2) = df_1 \times df_2$.

1.5 Immersions and Embeddings

Exercise 5.9

(1) Show that any parameterization $\varphi : U \subset \mathbb{R}^m \to M$ is an embedding of U into M.

(2) Show that, locally, any immersion is an embedding, i.e. if $f : M \to N$ is an immersion and $p \in M$, then there is an open set $W \subset M$ containing p such that $f|_W$ is an embedding.

(3) Let N be a manifold. Show that $M \subset N$ is a submanifold of N of dimension m if and only if, for each $p \in M$, there is a coordinate system $x : W \to \mathbb{R}^n$ around p on N, for which $M \cap W$ is defined by the equations $x^{m+1} = \cdots = x^n = 0$.

(4) Consider the sphere

$$S^n = \left\{ x \in \mathbb{R}^{n+1} \mid \left(x^1\right)^2 + \cdots + \left(x^{n+1}\right)^2 = 1 \right\}.$$

Show that S^n is an n-dimensional submanifold of $\mathbb{R}^{n+1}$ and that

$$T_x S^n = \left\{ v \in \mathbb{R}^{n+1} \mid \langle x, v \rangle = 0 \right\},$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^n$.

(5) Let $f : M \to N$ be a differentiable map between smooth manifolds and consider submanifolds $V \subset M$ and $W \subset N$. Show that if $f(V) \subset W$ then $f : V \to W$ is also a differentiable map.

(6) Let $f : M \to N$ be an injective immersion. Show that if M is compact then $f(M)$ is a submanifold of N.

1.6 Vector Fields

Exercise 6.11

(1) Let $X : M \to TM$ be a differentiable vector field on M and, for a smooth function $f : M \to \mathbb{R}$, consider its directional derivative along X defined by

$$\begin{aligned} X \cdot f : M &\to \mathbb{R} \\ p &\mapsto X_p \cdot f. \end{aligned}$$

Show that:

(a) $(X \cdot f)(p) = (df)_p X_p$;

(b) the vector field X is smooth if and only if $X \cdot f$ is a differentiable function for any smooth function $f : M \to \mathbb{R}$;

(c) the directional derivative satisfies the following properties: for $f, g \in C^\infty(M)$ and $\alpha \in \mathbb{R}$,
  (i) $X \cdot (f + g) = X \cdot f + X \cdot g$;
  (ii) $X \cdot (\alpha f) = \alpha(X \cdot f)$;
  (iii) $X \cdot (fg) = fX \cdot g + gX \cdot f$.

(2) Prove Proposition 6.3.

(3) Show that $(\mathbb{R}^3, \times)$ is a Lie algebra, where $\times$ is the cross product on $\mathbb{R}^3$.

(4) Compute the flows of the vector fields $X, Y, Z \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$X_{(x,y)} = \frac{\partial}{\partial x}; \quad Y_{(x,y)} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \quad Z_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

(5) Let $X_1, X_2, X_3 \in \mathfrak{X}(\mathbb{R}^3)$ be the vector fields defined by

$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, \quad X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \quad X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x},$$

where (x, y, z) are the usual Cartesian coordinates.

(a) Compute the Lie brackets $[X_i, X_j]$ for $i, j = 1, 2, 3$.
(b) Show that $\mathrm{span}\{X_1, X_2, X_3\}$ is a Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$, isomorphic to $(\mathbb{R}^3, \times)$.
(c) Compute the flows $\psi_{1,t}, \psi_{2,t}, \psi_{3,t}$ of X_1, X_2, X_3.
(d) Show that $\psi_{i,\frac{\pi}{2}} \circ \psi_{j,\frac{\pi}{2}} \neq \psi_{j,\frac{\pi}{2}} \circ \psi_{i,\frac{\pi}{2}}$ for $i \neq j$.

(6) Give an example of a non-complete vector field.

(7) Let N be a differentiable manifold, $M \subset N$ a submanifold and $X, Y \in \mathfrak{X}(N)$ vector fields tangent to M, i.e. such that $X_p, Y_p \in T_pM$ for all $p \in M$. Show that $[X, Y]$ is also tangent to M, and that its restriction to M coincides with the Lie bracket of the restrictions of X and Y to M.

(8) Let $f : M \to N$ be a smooth map between manifolds. Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be f-**related** (and we write $Y = f_*X$) if, for each $q \in N$ and $p \in f^{-1}(q) \subset M$, we have $(df)_pX_p = Y_q$. Show that:

(a) given f and X it is possible that no vector field Y is f-related to X;

(b) the vector field X is f-related to Y if and only if, for any differentiable function g defined on some open subset W of N, $(Y \cdot g) \circ f = X \cdot (g \circ f)$ on the inverse image $f^{-1}(W)$ of the domain of g;
(c) for differentiable maps $f : M \to N$ and $g : N \to P$ between smooth manifolds and vector fields $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ and $Z \in \mathfrak{X}(P)$, if X is f-related to Y and Y is g-related to Z, then X is $(g \circ f)$-related to Z.

(9) Let $f : M \to N$ be a diffeomorphism between smooth manifolds. Show that $f_*[X, Y] = [f_*X, f_*Y]$ for every $X, Y \in \mathfrak{X}(M)$. Therefore, f_* induces a Lie algebra isomorphism between $\mathfrak{X}(M)$ and $\mathfrak{X}(N)$.

(10) Let $f : M \to N$ be a differentiable map between smooth manifolds and consider two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Show that:

(a) if the vector field Y is f-related to X then any integral curve of X is mapped by f into an integral curve of Y;
(b) the vector field Y is f-related to X if and only if the local flows F_X and F_Y satisfy $f(F_X(p, t)) = F_Y(f(p), t)$ for all (t, p) for which both sides are defined.

(11) (*Lie derivative of a function*) Given a vector field $X \in \mathfrak{X}(M)$, we define the **Lie derivative** of a smooth function $f : M \to \mathbb{R}$ in the direction of X as

$$L_X f(p) := \frac{d}{dt}\left((f \circ \psi_t)(p)\right)_{|t=0},$$

where $\psi_t = F(\cdot, t)$, for F the local flow of X at p. Show that $L_X f = X \cdot f$, meaning that the Lie derivative of f in the direction of X is just the directional derivative of f along X.

(12) (*Lie derivative of a vector field*) For two vector fields $X, Y \in \mathfrak{X}(M)$ we define the **Lie derivative** of Y in the direction of X as

$$L_X Y := \frac{d}{dt}((\psi_{-t})_* Y)_{|t=0},$$

where $\{\psi_t\}_{t \in I}$ is the local flow of X. Show that:

(a) $L_X Y = [X, Y]$;
(b) $L_X[Y, Z] = [L_X Y, Z] + [Y, L_X Z]$, for $X, Y, Z \in \mathfrak{X}(M)$;
(c) $L_X \circ L_Y - L_Y \circ L_X = L_{[X,Y]}$.

(13) Let $X, Y \in \mathfrak{X}(M)$ be two complete vector fields with flows ψ, ϕ. Show that:

(a) given a diffeomorphism $f : M \to M$, we have $f_* X = X$ if and only if $f \circ \psi_t = \psi_t \circ f$ for all $t \in \mathbb{R}$;

(b) $\psi_t \circ \phi_s = \phi_s \circ \psi_t$ for all $s, t \in \mathbb{R}$ if and only if $[X, Y] = 0$.

1.7 Lie Groups

Exercise 7.17

(1) (a) Given two Lie groups G_1, G_2, show that $G_1 \times G_2$ (the direct product of the two groups) is a Lie group with the standard differentiable structure on the product.

(b) The circle S^1 can be identified with the set of complex numbers of absolute value 1. Show that S^1 is a Lie group and conclude that the n-torus $T^n \cong S^1 \times \ldots \times S^1$ is also a Lie group.

(2) (a) Show that $(\mathbb{R}^n, +)$ is a Lie group, determine its Lie algebra and write an expression for the exponential map.

(b) Prove that, if G is an abelian Lie group, then $[V, W] = 0$ for all $V, W \in \mathfrak{g}$.

(3) We can identify each point in

$$H = \left\{(x, y) \in \mathbb{R}^2 \mid y > 0\right\}$$

with the invertible affine map $h : \mathbb{R} \to \mathbb{R}$ given by $h(t) = yt + x$. The set of all such maps is a group under composition; consequently, our identification induces a group structure on H.

(a) Show that the induced group operation is given by

$$(x, y) \cdot (z, w) = (yz + x, yw),$$

and that H, with this group operation, is a Lie group.

(b) Show that the derivative of the left translation map $L_{(x,y)} : H \to H$ at a point $(z, w) \in H$ is represented in the above coordinates by the matrix

$$\left(dL_{(x,y)}\right)_{(z,w)} = \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}.$$

Conclude that the left-invariant vector field $X^V \in \mathfrak{X}(H)$ determined by the vector

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \in \mathfrak{h} \equiv T_{(0,1)} H \qquad (\xi, \eta \in \mathbb{R})$$

is given by

$$X^V_{(x,y)} = \xi y \frac{\partial}{\partial x} + \eta y \frac{\partial}{\partial y}.$$

(c) Given $V, W \in \mathfrak{h}$, compute $[V, W]$.

(d) Determine the flow of the vector field X^V, and give an expression for the exponential map $\exp : \mathfrak{h} \to H$.

(e) Confirm your results by first showing that H is the subgroup of $GL(2)$ formed by the matrices

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

with $y > 0$.

(4) Consider the group

$$SL(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \right\},$$

which we already know to be a 3-manifold. Making

$$a = p + q, \; d = p - q, \; b = r + s, \; c = r - s,$$

show that $SL(2)$ is diffeomorphic to $S^1 \times \mathbb{R}^2$.

(5) Give examples of matrices $A, B \in \mathfrak{gl}(2)$ such that $e^{A+B} \neq e^A e^B$.

(6) For $A \in \mathfrak{gl}(n)$, consider the differentiable map

$$\begin{aligned} h : \mathbb{R} &\to \mathbb{R}\backslash\{0\} \\ t &\mapsto \det e^{At} \end{aligned}$$

and show that:

(a) this map is a group homomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash\{0\}, \cdot)$;
(b) $h'(0) = \mathrm{tr} A$;
(c) $\det(e^A) = e^{\mathrm{tr} A}$.

(7) (a) If $A \in \mathfrak{sl}(2)$, show that there is a $\lambda \in \mathbb{R} \cup i\mathbb{R}$ such that

$$e^A = \cosh \lambda \; I + \frac{\sinh \lambda}{\lambda} A.$$

(b) Show that $\exp : \mathfrak{sl}(2) \to SL(2)$ is not surjective.

(8) Consider the vector field $X \in \mathfrak{X}(\mathbb{R}^2)$ defined by

$$X = \sqrt{x^2 + y^2} \frac{\partial}{\partial x}.$$

(a) Show that the flow of X defines a free action of $\mathbb{R}$ on $M = \mathbb{R}^2 \setminus \{0\}$.
(b) Describe the topological quotient space $M/\mathbb{R}$. Is the action above proper?

(9) Let $M = S^2 \times S^2$ and consider the diagonal S^1-action on M given by

$$e^{i\theta} \cdot (u, v) = \left(e^{i\theta} \cdot u, e^{2i\theta} \cdot v\right),$$

where, for $u \in S^2 \subset \mathbb{R}^3$ and $e^{i\beta} \in S^1$, $e^{i\beta} \cdot u$ denotes the rotation of u by an angle β around the z-axis.

(a) Determine the fixed points for this action.
(b) What are the possible nontrivial stabilizers?

(10) Let G be a Lie group and H a closed Lie subgroup, i.e. a subgroup of G which is also a closed submanifold of G. Show that the action of H in G defined by $A(h, g) = h \cdot g$ is free and proper.

(11) (*Grassmannian*) Consider the set $H \subset GL(n)$ of invertible matrices of the form

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

where $A \in GL(k)$, $B \in GL(n-k)$ and $C \in \mathcal{M}_{(n-k)\times k}$.

(a) Show that H is a closed Lie subgroup of $GL(n)$. Therefore H acts freely and properly on $GL(n)$ [cf. Exercise 7.17(10)].
(b) Show that the quotient manifold

$$Gr(n,k) := GL(n)/H$$

can be identified with the set of k-dimensional subspaces of $\mathbb{R}^n$ (in particular $Gr(n,1)$ is just the projective space $\mathbb{R}P^{n-1}$).
(c) The manifold $Gr(n,k)$ is called the **Grassmannian** of k-planes in $\mathbb{R}^n$. What is its dimension?

(12) Let G and H be Lie groups and $F : G \to H$ a Lie group homomorphism. Show that:

(a) $(dF)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism;
(b) if $(dF)_e$ is an isomorphism then F is a local diffeomorphism;
(c) if F is a surjective local diffeomorphism then F is a covering map.

(13) (a) Show that $\mathbb{R}\cdot SU(2)$ is a four-dimensional real linear subspace of $\mathcal{M}_{2\times 2}(\mathbb{C})$, closed under matrix multiplication, with basis

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

$$j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

satisfying $i^2 = j^2 = k^2 = ijk = -1$. Therefore this space can be identified with the **quaternions** (cf. Sect. 1.10.1). Show that $SU(2)$ can be identified with the quaternions of Euclidean norm equal to 1, and is therefore diffeomorphic to S^3.

(b) Show that if $n \in \mathbb{R}^3$ is a unit vector, which we identify with a quaternion with zero real part, then

$$\exp\left(\frac{n\theta}{2}\right) = 1\cos\left(\frac{\theta}{2}\right) + n\sin\left(\frac{\theta}{2}\right)$$

is also a unit quaternion.

(c) Again identifying $\mathbb{R}^3$ with quaternions with zero real part, show that the map

$$\begin{aligned} \mathbb{R}^3 &\to \mathbb{R}^3 \\ v &\mapsto \exp\left(\frac{n\theta}{2}\right) \cdot v \cdot \exp\left(-\frac{n\theta}{2}\right) \end{aligned}$$

is a rotation by an angle θ about the axis defined by n.

(d) Show that there exists a surjective homomorphism $F : SU(2) \to SO(3)$, and use this to conclude that $SU(2)$ is the universal covering of $SO(3)$.

(e) What is the fundamental group of $SO(3)$?

1.8 Orientability

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1.9 Manifolds with Boundary

Exercise 9.5

(1) Show with an example that the product of two manifolds with boundary is not always a manifold with boundary.
(2) Let M be a manifold without boundary and N a manifold with boundary. Show that the product $M \times N$ is a manifold with boundary. What is $\partial(M \times N)$?
(3) Show that a diffeomorphism between two manifolds with boundary M and N maps the boundary ∂M diffeomorphically onto ∂N.

後面有 Notes 是各節的補充說明。