Geometry and Topology Qualifying Examination
Fall 2020

October 30, 2020

1. Let (M^n, g) be a Riemannian manifold.
   (a) Define the Levi-Civita connection on M. (5%)
   (b) Let $p \in M$ and U be a local chart of p. Suppose $(x_1, \cdots, x_0)$ is the local coordinate on U. Define the Christoffel symbols $\Gamma_{ij}^k (i, j, k = 1, \cdots, n)$ by
   $$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_i} = \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$
   Show that $\Gamma_{ij}^k = \Gamma_{ji}^k$. (10%)
   (c) Define the Riemann curvature $R(X, Y)Z$ for any $X, Y, Z \in TM$. (5%)
   (d) State and prove the Bianchi Identity of the Riemann curvature tensor R. (10%)

2. Let M be a Riemannian manifold and $f \in C^3(M)$. Suppose $\{x_i\}_{i=1}^n$ is a normal coordinate system at $p \in M$. Derive the following Bochner formula:
   $$\frac{1}{2}\Delta|\nabla f|^2 = \sum_{i,j} |f_{ij}|^2 + \sum_{i,j} R_{ij} f_i f_j + \sum_i f_i (\Delta f)_i$$
   where f_i denotes the derivative of f with respect to $\frac{\partial}{\partial x_i}$ and R_{ij} is the Ricci curvature. (15%)

3. Let $M^2 \subseteq \mathbb{R}^3$ be an embedded compact, closed surface of genus ≥ 1. Show that the Gaussian curvature of M must be vanish somewhere on M. (15%)

4. Consider the torus of revolution T obtained by rotating the circle $(x - a)^2 + Z^2 = r^2$ around z-axis:
   $$T = \{(x, y, z) | (x^2 + y^2 + z^2 + a^2 - r^2)^2 - 4a^2(x^2 + y^2) = 0\}$$
   Parametrize this torus, compute its Gaussian curvature function K, and verify $\int_T K dA = 0$ by explicit calculation. (20%)

5. Let (M, g) be a Riemannian manifold. Let $p \in M$ and the map $exp_p : B_\varepsilon(0) \subset T_pM \to M$ is the exponential map at p which is always defined in a small neighborhood $B_\varepsilon(0)$ of the origin of T_pM. Show that there exists a $\delta > 0$ such that
   $$exp_p : B_\delta(0) \subset T_pM \to M$$
   is a diffeomorphism onto its image. (10%)

6. Show that there does not exist any nonconstant harmonic function on a compact, connected Riemannian manifold without boundary. (10%)

Geometry and Topology Qualifying Examination
Spring 2021

April 28, 2021

1. Let ∇ be the Levi-Civita connection on a Riemannian n-manifold M with a metric g_{ij} defined by

$$g_{ij} = \langle \tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j} \rangle, \quad i, j = 1, \cdots, n.$$

Define the Christoffel symbol Γ_{ij}^k by

$$\nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_j} = \Gamma_{ij}^k \tfrac{\partial}{\partial x_k}, \quad i, j, k = 1, \cdots, n.$$

and the Riemannian curvature tensor R_{ijl}^m by

$$Rm(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j})\tfrac{\partial}{\partial x_l} = R^m{}_{ijl}\tfrac{\partial}{\partial x_m} \quad \text{and} \quad R_{ijkl} = g_{mk}R^m{}_{ijl}.$$

Here

$$Rm(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j})\tfrac{\partial}{\partial x_l} = \nabla_{\frac{\partial}{\partial x_i}}\nabla_{\frac{\partial}{\partial x_j}}\tfrac{\partial}{\partial x_l} - \nabla_{\frac{\partial}{\partial x_j}}\nabla_{\frac{\partial}{\partial x_i}}\tfrac{\partial}{\partial x_l}.$$

Finally we define the Ricci curvature tensor and scalar curvature by

$$R_{ij} = g^{kl}R_{ikjl} \;\; \textit{and} \;\; R = g^{ij}R_{ij}.$$

(a) (5%)Show that

$$\Gamma_{ij}^k = \Gamma_{ji}^k.$$

(b) (5%)Show that

$$\Gamma_{ij}^k = \tfrac{1}{2}g_{kl}(g_{lj,i} + g_{il,j} - g_{ij,l}). \quad \text{Here} \quad g_{ij,k} = \tfrac{\partial g_{ij}}{\partial x_k}.$$

(c) (5%)Show that

$$R^m{}_{ijl} = \tfrac{\partial}{\partial x_i}\Gamma_{jl}^m - \tfrac{\partial}{\partial x_j}\Gamma_{il}^m + \Gamma_{in}^m\Gamma_{jl}^m - \Gamma_{jn}^m\Gamma_{il}^m.$$

(d) (5%)Show that

$$\nabla_{\frac{\partial}{\partial x_i}} g_{jk} = 0.$$

(e) (10%)State and prove Branchi identity.

2. Let $(R^2, g(t))$ be a complete Riemannian surface with

$$g(t) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}, \quad dx^2 = dx \bigotimes dx.$$

   (a) (10%) Show that in polar coordinates (r, θ), we may rewrite

   $$g(0) = ds^2 + \tan h^2 s d\theta^2, \quad s = \log(r + \sqrt{1+r^2}).$$

   (b) (10%) Show that the scalar curvature R_0 of (R^2, g_0), $g_0 := g(0)$, satisfying $R_0 = \frac{4}{1+r^2}$.

3. (25%) Let M be an m-dimensional complete Riemannian manifold with

$$R_{ij} \geq (m-1)K \text{ for some } K > 0.$$

   Show that M is a compact manifold with diameter less than $\frac{\pi}{\sqrt{K}}$.

4. (25%) For disk $M = \{(x, y) \in R^2 | x^2 + y^2 < 1\}$ with the metric

$$ds^2 = \frac{4}{(1-x^2-y^2)^2}(dx^2 + dy^2).$$

   Show that M is a complete Riemanifold with sectional curvature being -1 everywhere.