§ On a Remannian manifold (M, g), a vector field U is called a Killing vector field if it is infinitesimally an isometry, namely $\frac{d}{dt}\Big|_{t=0} \varphi_t^* g = 0$

Where φ_t is the one-paremeter family of diffeomorphism generate by U

(1) For a Killing vector field U, show that $g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0$ for any two vector fields X,Y

(2) Suppose that U is a Killing vector field, and γ is a geodesic。Prove that $U\big|_\gamma$ is a Jacobi field

Proof: Compute:

$$R(\gamma', \xi)\gamma' = (\nabla_{\gamma'}(\nabla \xi))\gamma' = \nabla_{\gamma'}\nabla_{\gamma'}\xi - \nabla_{\nabla_{\gamma'}\gamma'}\xi = \frac{D^2\xi}{dt^2},$$

using that $\nabla_{\gamma'}\gamma' = \mathbf{0}$.

§ Killing vector field

Let (M g) be a Riemannian manifold with Levi-Civita connection ∇, and let $\psi_t : M \to M$ be a 1-parameter group of isometries。

The vector field $X \in \chi(M)$ defined by

$X_p := \frac{d}{dt}\Big|_{t=0} \psi_t(p)$ is called Killing vector field associated to ψ_t

或者直接以 $\mathrm{L}_X g = 0$ 為定義。

§ Jacobi field

$\gamma:[0,l] \to \mathrm{M}$ is a geodesic。

J(t) is a vector field J(t) along γ, satisfies

$\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0$ for all $t \in [0, l]$

Then J(t) is called a Jacobi field。

或者寫成：

$\nabla_T^2 J + R(J,T)T = 0$，其中 $T = \frac{d\gamma}{dt}$，($\nabla_T^2 J$ 即 $\frac{D^2 J}{dt^2}$)

其中 $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ 為曲率張量