1. $r=1+\cos\theta$ is a cardioid(心臟線)。At the point $(r,\theta)=(2,0)$，

(1) find the curvature

The formula for curvature in polar coordinates is：

$$\kappa=\frac{r^2+2(\frac{dr}{d\theta})^2-r\frac{d^2r}{d\theta^2}}{(r^2+(\frac{dr}{d\theta})^2)^{3/2}}$$ ，evaluate at $r=2,\theta=0$

Then $\kappa=\frac{3}{4}$

y=f(x)的 curvature $\kappa=\frac{d\theta}{ds}=\frac{d\theta}{dx}\cdot\frac{dx}{ds}=\frac{f''(x)}{[1+(f'(x)^2]^{\frac{3}{2}}}$

or $\kappa=\frac{\left|\dot{X}\wedge\ddot{X}\right|}{(\dot{X}\cdot\dot{X})^{3/2}}$ where $\mathrm{X}(t)=(x(t),y(t),z(t))$ $T=\frac{dX}{ds}$ then $\kappa=\left|\frac{dT}{ds}\right|$

Now that $X=((1+\cos\theta)\cos\theta,(1+\cos\theta)\sin\theta)$

$$\dot{X}=(-\sin\theta-\sin 2\theta,\cos\theta+\cos 2\theta)$$

$$\ddot{X}=(-\cos\theta-2\cos 2\theta,-\sin\theta-2\sin 2\theta)$$

$\theta=0$ $\kappa=\ldots=\frac{6}{8}=\frac{3}{4}$

(2) If we consider this cardioid as a surface of revolution about z-axis，find the Gaussian curvature K

$x=r\cos\theta=\cos\theta(1+\cos\theta)$

$y=r\sin\theta=\sin\theta(1+\cos\theta)$

If we consider this cardioid as a surface of revolution about z-axis，then its embedding in R^3 is given by

$X(\theta,\phi)=((1+\cos\theta)\cos\phi,(1+\cos\theta)\sin\phi,\theta)$

Where θ is the parameter from the polar equatio，ϕ is the revolution angle around the z-axis。

The Gaussian curvature is given by $K=\frac{\det(II)}{\det(I)}$ ，K(2,0)=0

2. $\frac{x^2}{9}+\frac{y^2}{4}+\frac{z^2}{1}=1$ is an ellipsoid。At the point (x,y,z)=(3,0,0)，Gauss curvature K=

For a general ellipsoid of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

The Gaussian curvature at point (x_0, y_0, z_0) is given by： $K=\frac{eg-f^2}{EG-F^2}$

The implicit surface formula for Gaussian curvature：

$$K=-\frac{\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & F_x \\ F_{yx} & F_{yy} & F_{yz} & F_y \\ F_{zx} & F_{zy} & F_{zz} & F_z \\ F_x & F_y & F_z & 0 \end{vmatrix}}{(F_x{}^2+F_y{}^2+F_z{}^2)^2}$$

$\mathrm{F}(x,y,z)=\frac{x^2}{9}+\frac{y^2}{4}+\frac{z^2}{1}-1$， $\nabla F=(\frac{2x}{9},\frac{y}{2},2z)$

$HessF=\begin{pmatrix} \frac{2}{9} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 分子(at (3,0,0))=$\frac{4}{9}$ ，分母(at (3,0,0))=$(\frac{4}{9})^2$

$K=\frac{9}{4}$

DeepSeek: $\frac{9}{4}$

1. **Parametrization of the Ellipsoid**:
   The ellipsoid can be parametrized using modified spherical coordinates:

$$\mathbf{r}(\theta,\phi)=(3\sin\phi\cos\theta, 2\sin\phi\sin\theta, \cos\phi)$$

   The point $(3,0,0)$ corresponds to $\theta=0$ and $\phi=\frac{\pi}{2}$.

2. **First Fundamental Form:**

- Compute the partial derivatives $\mathbf{r}_\theta$ and $\mathbf{r}_\phi$ at $\theta = 0$ and $\phi = \frac{\pi}{2}$:

$$\mathbf{r}_\theta = (0,2,0), \quad \mathbf{r}_\phi = (0,0,-1)$$

- Coefficients of the first fundamental form:

$$E = 4, \quad F = 0, \quad G = 1$$

- The determinant $EG - F^2 = 4 \times 1 - 0 = 4$.

3. **Second Fundamental Form:**

- Compute the second partial derivatives and the unit normal vector:

$$\mathbf{r}_{\theta\theta} = (-3,0,0), \quad \mathbf{r}_{\theta\phi} = (0,0,0), \quad \mathbf{r}_{\phi\phi} = (-3,0,0)$$

The unit normal vector $\mathbf{N}$ is $(-1,0,0)$.

- Coefficients of the second fundamental form:

$$L = 3, \quad M = 0, \quad N = 3$$

- The determinant $LN - M^2 = 3 \times 3 - 0 = 9$.

ChatGPT：$\dfrac{9}{4}$

3. $\Gamma{:}y = x^2$ is a parabola， $ds = \sqrt{dx^2 + dy^2}$ ， κ is the curvature，then $\int_\Gamma \kappa ds =$

κ is ginven by $\kappa = \dfrac{|y''|}{(1+(y')^2)^{3/2}}$ ，y”=2， $\kappa = \dfrac{2}{(1+4x^2)^{3/2}}$

$ds = \sqrt{1+4x^2}dx$

$$\int \kappa ds = \frac{2}{(1+4x^2)^{3/2}} \cdot \sqrt{1+4x^2}dx = \int \frac{2}{1+4x^2}dx$$

Note that $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$ ，we obtain $\int \dfrac{2}{1+4x^2}dx = \tan^{-1}(2x)$

$$\int_{-\infty}^{\infty} \kappa ds = \pi$$

The **total curvature** of the parabola is $\int_\Gamma \kappa ds = \pi$

4. $z = \dfrac{x^2}{4} + y^2$ is an elliptic parabolid。x-y+2z=4 intesects this paraboloid in a planar

curve ，At the point (2,0,1)，the curvature K= ? $K = \frac{37\sqrt{6}}{242}$

Answer by DeepSeek

Parametrize the Curve: Substitute $z = \frac{x^2}{4} + y^2$ into the plane equation $x - y + 2z = 4$:

- Simplify to get the quadratic equation: $\frac{x^2}{2} + 2y^2 + x - y = 4$.
- Complete the square to identify the ellipse: $\frac{(x+1)^2}{37/4} + \frac{(y-1/4)^2}{37/16} = 1$.

Parametrization: Use the standard parametrization of an ellipse:

- $x = -1 + \frac{\sqrt{37}}{2}\cos\theta$
- $y = \frac{1}{4} + \frac{\sqrt{37}}{4}\sin\theta$
- $z = \frac{x^2}{4} + y^2$

4. **Find θ Corresponding to the Point**: Solve for θ at $(2, 0, 1)$:
   - $\cos\theta = \frac{6}{\sqrt{37}}$ and $\sin\theta = -\frac{1}{\sqrt{37}}$.
5. **Compute Derivatives**: Calculate the first and second derivatives of the parametrization with respect to θ.
6. **Cross Product and Magnitude**: Use the curvature formula for a 3D curve:
   - $\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$
   - Compute the cross product and its magnitude, then the magnitude of the first derivative vector.

7. **Final Calculation**: Substitute the values into the curvature formula:
   - $|\mathbf{r}' \times \mathbf{r}''| = \frac{37}{16}\sqrt{6}$
   - $|\mathbf{r}'| = \frac{\sqrt{11}}{2}$
   - $|\mathbf{r}'|^3 = \left(\frac{\sqrt{11}}{2}\right)^3 = \frac{11^{3/2}}{8}$
   - Curvature $\kappa = \frac{\frac{37}{16}\sqrt{6}}{\frac{11^{3/2}}{8}} = \frac{37\sqrt{66}}{242}$

Answer by ChatGPT

The given equations are:

$$S_1 : z = \frac{x^2}{4} + y^2$$

$$S_2 : x - y + 2z = 4$$

The curve lies at the intersection of these two surfaces. We take the gradients of the surfaces to find their normal vectors.

Compute ∇S_1 (Gradient of S_1)

$$\nabla S_1 = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right) = \left(\frac{x}{2}, 2y, -1\right)$$

At (2,0,1) $\nabla S_1(2,0,1) = (1,0,-1)$

Rewriting S_2 as:

$$g(x, y, z) = x - y + 2z - 4 = 0$$

$$\nabla S_2 = (1, -1, 2)$$

At $(2, 0, 1)$:

$$\nabla S_2(2, 0, 1) = (1, -1, 2)$$

The tangent direction to the intersection curve is given by the cross product:

$$\mathbf{T} = \nabla S_1 \times \nabla S_2$$

Computing the determinant:

$$\mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

=(-1,-1,-1)

Curvature is given by:

$$\kappa = \left| \frac{d\mathbf{T}/ds}{ds} \right|$$

$\kappa = 0$

5. Cone $C = \{(x,y,z) | x^2 + y^2 = z^2\}$ ， $E: x - 2y + 3z = 4$ ， $C \cap E$ is a conic section。

At (x,y,z)= (-3,4,5)，the curvature $\kappa = ?$

ChatGPT： $\frac{1}{10}$

DeepSeek： $\frac{\sqrt{21}}{9}$

兩者都是$\kappa = \frac{|r' \times r''|}{|r'|^3}$