On R^3，consder the 1-form $\omega = dz + \frac{1}{2}(xdy - ydx)$

(a) Calculate $d\omega$ and $\omega \wedge d\omega$

(b) Note that kernel ω is everywhere 2-dimensional。

Check that $d\omega|_{\ker\omega}$ is everywhere non-degenerate

(c) Suppose that U and V are vector fields defined on the unite ball B，which are pointwise linearly independent and belongs to the kernel of ω。
Prove that [U,V] is nowhere in the kernel of ω。

$d\omega = dx \wedge dy$ ， $\omega \wedge d\omega = dx \wedge dy \wedge dz$

The kernel of a differential one-form： $Ker(\omega) := \{V \in \chi(M) | \omega(V) = 0\}$

Check that $d\omega$ restricted to $\ker(\omega)$ is everywhere **non-degenerate**

Let $V = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ ，then $\omega(V) = 0$ gives $c + \frac{1}{2}(xb - ya) = 0$

This provides a constraint，reducing the dimension of $\ker(\omega)$ to two，confirming the claim that $\ker(\omega)$ is 2-dimensional。

To check $d\omega$ (on $\ker(\omega)$) is non-degenerate，we need to verify that the restriction of $d\omega$ to $\ker(\omega)$ is a symplectic form，meaning that it remains non-degenerate (i.e., its determinant is nonzero in local coordinates).

The $\ker\omega$ consists of vector field satisfying $c = -\frac{1}{2}(xb - ya)$ ，so it is spanned by two independent vectors，say ：

$V_1 = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, V_2 = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}$ $(V = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ let a=1,b=0 or a=0,b=1)

Compute $d\omega(V_1, V_2)$

Since $dx(V_1) = 1, dy(V_1) = 0, dx(V_2) = 0, dy(V_2) = 1$

$$d\omega(V_1, V_2) = dx \wedge dy(V_1, V_2) = \begin{vmatrix} dx(V_1) & dx(V_2) \\ dy(V_1) & dy(V_2) \end{vmatrix} = 1 \neq 0$$

$d\omega$ is non-degenerate on $\ker\omega$ ，confirming that it defines a symplectic structure on the 2-dimensional distribution。

A symplectic form：

A symplectic form on a smooth manifold M is a 2-form ω (a differential form of degree 2) that satisfies the following properties:

1. **Closedness**: The exterior derivative of ω is zero, i.e., $d\omega = 0$. This means ω is a closed form.
2. **Non-degeneracy**: For every point $p \in M$ and every non-zero tangent vector $v \in T_pM$, there exists another tangent vector $w \in T_pM$ such that $\omega_p(v, w) \neq 0$. This implies that ω defines a non-degenerate bilinear form on each tangent space.

A manifold M equipped with a smyplectic form ω is called a symplectic manifold。
A symplectic forms are used to decribe the phase space of mechanical systems。

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

Example

Let $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ ， $X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial y}$

$$\omega(Y) = \omega(\frac{\partial}{\partial y}) = xdz - zdx$$

$$\omega(X) = \omega(\frac{\partial}{\partial x}) = ydz - zdy$$

$$X(\omega(Y)) = \frac{\partial}{\partial x}(xdz - zdx) = dz$$

$$Y(\omega(X)) = \frac{\partial}{\partial y}(ydz - zdy) = dz$$

Lie bracket $[X,Y] = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ ， $\omega[X,Y] = 0$

Then $d\omega(X,Y) = 0$

A computation of $\omega(X,Y) =$ where ω is a 2-form：

$\omega = dx \wedge dy$

$X = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial z}, Y = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial z}$ compute $\omega(X,Y) =$

$$\omega(X,Y) = \omega_{ij} X^i Y^j$$

$$dx \wedge dy(\text{X,Y}) = \begin{vmatrix} dx(X) & dx(Y) \\ dy(X) & dy(Y) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Exercise

$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$

$X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ verifies that $\omega(X,Y) = x - y + z$