§ Lie Derivative [DF002]

$L_X S=\frac{d}{dt}(\varphi_t^* S)\big|_{t=0}$

1. $f:M\to R \quad L_X f=df(x)=X(f)$
2. $L_X Y=[X,Y]$
3. $\omega=\omega_j dx^j, X=X^i\frac{\partial}{\partial x^i}$ then $L_X\omega=(X^i\frac{\partial\omega_j}{\partial x^i}+\omega_i\frac{\partial X^i}{\partial x^j})dx^j$

   Cartan formula $L_X\omega=\iota_X(d\omega)+d(\iota_X\omega)$
4. η is a p-form then $L_X\eta=\iota(X)(d\eta)+d(\iota(X)\eta)$

[N3401LieDerivative] [Killing002S^2]

例 1 R^2 中

$\omega=ydx+xdy, X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$

$\iota_X\omega=xy+yx=2xy$ ， $d(\iota_X\omega)=2ydx+2xdy$

$d\omega=dy\wedge dx+dx\wedge dy=0$

所以 $L_X\omega=2ydx+2xdy$

例 2 S^2 中

$\omega=\sin^2\theta d\varphi, X=\partial\varphi$

$\iota_X\omega=\sin^2\theta$ ， $d(\iota_X\omega)=2\sin\theta\cos\theta d\theta$

$d\omega=2\sin\theta\cos\theta d\theta\wedge d\varphi$ ， $\iota_X(d\omega)=0\times d\varphi-2\sin\theta\cos\theta d\theta=-2\sin\theta\cos\theta d\theta$

$\iota_X(dx\wedge dy)=X^x dy-X^y dx$

所以 $L_X\omega=0$

例 3 R^2 中

$\omega=ydx-xdy, X=x^2\partial_x+xy\partial_y$

解 1 Cartan formula

$\iota_X\omega=x^2y-x^2y=0$ ，

$d\omega=dy\wedge dx-dx\wedge dy=-2dx\wedge dy$ ，

$\iota_X(d\omega)=-2x^2dy+2xydx=2x(ygx-xdy)=2x\omega$

解 2

$(L_X\omega)_i=X^j\partial_j\omega_i+\omega_j\partial_i X^j$ 　把 X 與 ω 的分量排好 方便計算

	x 分量	y 分量
X	x^2	xy
ω	y	-x

$$(L_X\omega)_x = X^x\partial_x\omega_x + \omega_x\partial_x X^x + X^y\partial_y\omega_x + \omega_y\partial_x X^y$$

$$= 0 + y(2x) + xy - xy = 2xy$$

$$(L_X\omega)_y = X^x\partial_x\omega_y + \omega_x\partial_y X^x + X^y\partial_y\omega_y + \omega_y\partial_y X^y$$

$$= x^2\times(-1) + y\times 0 + xy\times 0 + (-x)\times x = -2x^2$$

所以 $L_X\omega = 2xydx - 2x^2dy = 2x(ydx - xdy)$

例 4　S^2 中，$L_X g = 0$　for a Killing vector field

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial}{\partial\varphi}\text{，}Y = \frac{\partial}{\partial\theta} = \cos\theta\cos\varphi\frac{\partial}{\partial x} + \cos\theta\sin\varphi\frac{\partial}{\partial y} - \sin\theta\frac{\partial}{\partial z}$$

$$g = d\theta^2 + \sin^2\theta d\varphi^2$$

$$(L_X g)_{\mu\nu} = X^\rho\partial_\rho g_{\mu\nu} + \partial_\mu X^\rho g_{\rho\nu} + \partial_\nu X^\rho g_{\rho\mu}$$

經過小心計算結果：

$(L_X g)_{\mu\nu} = 0, (L_Y g)_{22} = 2\sin\theta\cos\theta$，所以 X 是 Killing field，Y 不是。

用 Leibnitz 法則比較好算

$$L_Y g = L_Y(d\theta\otimes d\theta) + L_Y(\sin^2\theta d\phi\otimes d\phi)$$

$$= (L_Y d\theta)\otimes d\theta + d\theta\otimes(L_Y d\theta) + (L_Y\sin^2\theta)d\phi\otimes d\phi + \sin^2\theta(L_Y d\phi)\otimes d\phi + \sin^2\theta d\phi\otimes(L_Y d\phi)$$

$$L_Y d\theta = 0, L_Y d\phi = 0$$

所以 $L_Y g = 2\sin\theta\cos\theta d\phi^2$