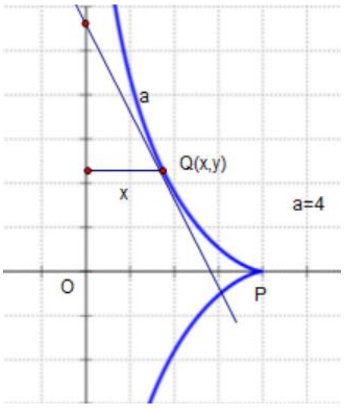
§ 曳物線 tractrix



開始時，物體在 P(a,0)，拖曳者在原點

拖譯者往 y 軸正向移動，使得每一刻拖線與拖曳線相切，所以 $\frac{dy}{dx}=\pm\frac{\sqrt{a^2-x^2}}{x}$

這是一個 ODE，y(0)=a

$y(x)=\int_x^a \frac{\sqrt{a^2-t^2}}{t}dt$ ，with y(a)=0

$\int \frac{\sqrt{a^2-t^2}}{t}dt=$ let t=acosu，then dt=-asinudu

$$\int \frac{\sqrt{a^2-t^2}}{t}dt=-a\int \tan u\sin u du=-a(\ln|\sec u+\tan u|-\sin u)$$

$$=-a(\ln\left|\frac{a}{t}+\frac{\sqrt{a^2-t^2}}{t}\right|-\frac{\sqrt{a^2-t^2}}{a})+C$$

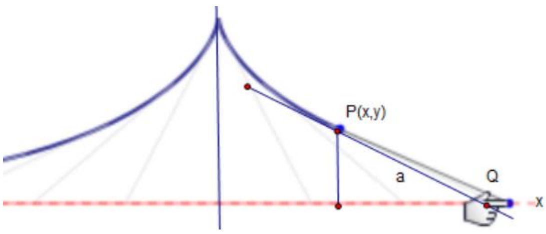
$$y(x)=\int_x^a \frac{\sqrt{a^2-t^2}}{t}dt=a(\ln\left|\frac{a}{t}+\frac{\sqrt{a^2-t^2}}{t}\right|-\frac{\sqrt{a^2-t^2}}{a})\Big|_a^x$$

$$=(a\ln\frac{a+\sqrt{a^2-x^2}}{x})-\sqrt{a^2-x^2}$$

where $\int \sec u du=\ln|\sec u+\tan u|+C$

Let x=asint，then $\frac{a+\sqrt{a^2-x^2}}{x}=\ldots=\cot\frac{t}{2}$

寫成參數式，曳物線方程式為 $\begin{cases} x=a\sin t \\ y=a\ln\left|\cot\frac{t}{2}\right|-a\cos t \end{cases}$



另一種寫法：

$$\begin{cases} y=a\sin t \\ \frac{dy}{dx}=\tan t \end{cases} \Rightarrow \begin{cases} \dot{y}=a\cos t \\ \dot{x}=\dot{y}\cot t \end{cases}$$

$\dot{x} = \dot{y}\cot t = a\cos t\cot t = -a\sin t + \dfrac{a}{\sin t}$ 積分

$x = a\cos t + \int \dfrac{adt}{\sin t} = a\cos t + a\ln\left|\tan\dfrac{t}{2}\right| + C$ 得到：

$$\begin{cases} x = a\cos t + a\ln\left|\tan\dfrac{t}{2}\right| \\ y = a\sin t \end{cases}，0 \le t < \pi$$

Where $\int \dfrac{dt}{\sin t} = \ln\left|\tan\dfrac{t}{2}\right| + C$

查表 $\int \csc ax dx = -\dfrac{1}{a}\ln\left|\csc ax + \cot ax\right| + C$

It is easy to check that $\csc x + \cot x = \cot\dfrac{x}{2}$

1. Catenary as evolute of a tractrix