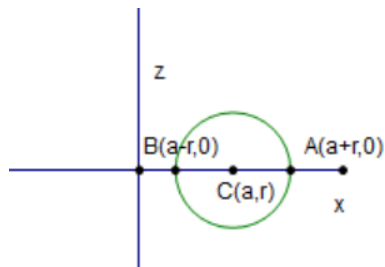


## § 環面 (torus)

Differential Geometry of Curves and Surfaces M.P.do Carmo p.260



Consider the torus of revolution generated by rotating the circle

$$(x-a)^2 + z^2 = r^2, y=0 \text{ about the } z \text{ axis } (a>r>0) .$$

The parallels generated by the points  $(a+r,0)$ ,  $(a-r,0)$ ,  $(a,r)$  are called the maximum parallel, the minimum parallel, and the upper parallel, respectively.

Check which of these parallels is

- (a) A geodesic
- (b) An asymptotic curve
- (c) A line of curvature
- (d) Compute the geodesic curvature of the upper parallel of the torus

(a) a geodesic C 參看 RG3301Geodesics

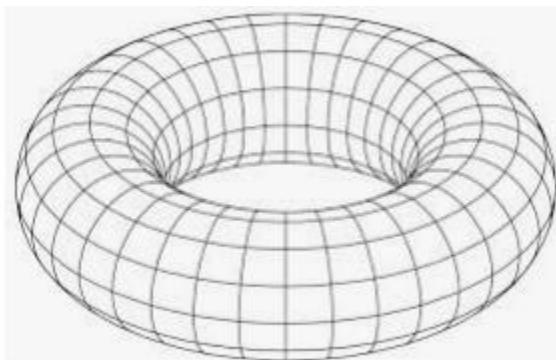
$$\forall p \in C, \kappa_g = 0, \frac{Dr'(t)}{dt} = 0 \text{ p.245}$$

(b) an asymptotic curve C

$$\forall p \in C, \kappa_n = 0, \frac{dt}{ds} = \kappa_n N + \kappa_g Y = \kappa_g Y$$

(c) a line of curvature C

$$\forall p \in C, \kappa_n \text{ is max or min normal curvature, } N'(t) = \lambda(t)\alpha'(t) \text{ p.145}$$



$$X(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin v)$$

$$0 < u < 2\pi, 0 < v < 2\pi$$

$$X_u = (-r \sin u \cos v, -r \sin u \sin v, r \cos u)$$

$$X_v = ((r \cos u + a)(-\sin v), (r \cos u + a) \cos v, 0)$$

$$X_u \times X_v = r(\cos u + a)(-\cos u \cos v, -\cos u \sin v, -\sin u)$$

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = (0, 0, -1) \quad , \text{ as } u = \frac{\pi}{2}$$

For the upper parallel  $\alpha(v) = (a \cos v, a \sin v, r)$

$$E = r^2, \quad F = 0, \quad G = (r \cos u + a)^2$$

$$\text{Now } u = \frac{\pi}{2}, \quad ds^2 = Edu^2 + Fdudv + Gdv^2 = a^2 dv^2 = a^2 dv^2$$

$$\alpha'(v) = \frac{d\alpha}{dv} = (-\sin v, \cos v, 0)$$

$$\alpha''(v) = \frac{d\alpha'}{ds} \frac{ds}{dv} = \frac{1}{a}(-\cos v, -a \sin v, 0) \quad \text{落在切平面上 所以}$$

$$\kappa_g = |\alpha''| = \frac{1}{a}, \quad \text{同時 } \alpha \text{ 是 asymptotic curve.}$$

另外兩個 parallels 的切向量由直觀為 principal direction

(即 normal curvature 為最大或最小, 故根據 line of curvature 定義為 line of curvature)

又 upper parallel 切向量  $v$  的 normal curvature = 0 ( $dN(v) = 0$ ) 同一點其他方向的 normal curvature 都為負 ( $= -\langle dN(v), v \rangle$  by the formula on page 142), 所以 upper parallel 切向量也是 principal direction, 故根據 line of curvature 定義 upper parallel 也是 line of curvature. ◦