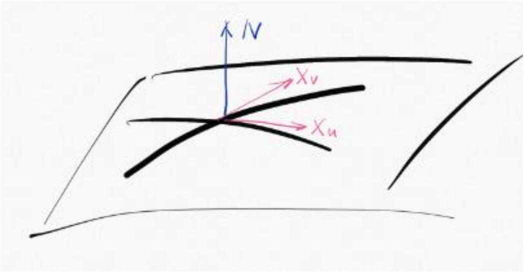

基本方程式

曲面 X=X(u,v)

Let $X_u = X_1, X_v = X_2$

$X_{ij} = \Gamma_{ij}^k X_k + b_{ij} N$

($b_{ij} = X_{ij} \cdot N$)

設 $[ij,k] = X_{ij} \cdot X_k$，則$[ij,k] = \frac{1}{2}\{\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ik}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\}$

A= $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij})$，則 $A^{-1} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix} = (g^{ij})$

引入 g^{jk}，使得 $g_{ij} g^{jk} = \delta_i^k$

$X_{ij} = \Gamma_{ij}^k X_k + b_{ij} N$ 兩邊對 X_l 做內積，再同乘 g^{kl}

則$\Gamma_{ij}^k = g^{kl}[ij,l]$

Gauss 方程

$X_{ij} = \Gamma_{ij}^k X_k + b_{ij} N$ ， $\Gamma_{ij}^k = [ij,l] g^{lk} = \frac{1}{2}\{\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{il}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k}\} g^{lk}$

Check $\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG-F)^2}$， $\Gamma_{ij}^k = \Gamma_{ji}^k$

......

$N_i = -(g^{kj} b_{ij}) X_k$ ......Weingarten 方程式

因為 $X_j \cdot N = 0$

$X_{ij} \cdot N + X_j \cdot N_i = 0$，let $N_i = p_{ik} X_k$ 兩邊對 X_j 作內積

Then $N_i \cdot X_j = -X_{ij} \cdot N = -b_{ij} = p_{ik} g_{jk}$，兩邊同乘

$p_{ik} = -b_{ij} g^{jk}$，所以 $N_i = (-g^{kj} b_{ij}) X_k$

$$\frac{\partial b_{ij}}{\partial u^l} - \Gamma_{il}^{k} b_{kj} = \frac{\partial b_{il}}{\partial u^j} - \Gamma_{ij}^{k} b_{kl}$$ ......Codazzi 方程式

$$(X_{ij})_l = (X_{il})_j$$

$$\frac{\partial}{\partial u^l}\left\{\Gamma_{ij}^{k} X_k + b_{ij} N\right\} = \frac{\partial}{\partial u^j}\left\{\Gamma_{il}^{k} X_k + b_{il} N\right\}$$

$$\frac{\partial \Gamma_{ij}^{k}}{\partial u^l} X_k + \Gamma_{ij}^{k} X_{kl} + \frac{\partial b_{ij}}{\partial u^l} N + b_{ij} N_l = \frac{\partial \Gamma_{il}^{k}}{\partial u^j} X_k + \Gamma_{il}^{k} X_{kj} + \frac{\partial b_{il}}{\partial u^j} N + b_{ij} N_j$$

兩邊對 N 作內積(取法部) 得

$\Gamma_{ij}^{k} b_{kl} + \frac{\partial b_{ij}}{\partial u^l} = \Gamma_{il}^{k} b_{kj} + \frac{\partial b_{il}}{\partial u^j}$,移項 得 Codazzi eq

Weingarten 方程

$$N_i = -(g^{kj} b_{ij}) X_k$$

Rodrigues 方程 $dN + \kappa dX = 0$ 是 lines of curvature 的微分式(其積分曲線即 lines of curvature)(Olinde Rodrigues)

測地線方程式

$$\frac{d^2 u^i}{ds^2} + \Gamma_{jk}^{i} \frac{du^j}{ds}\frac{du^k}{ds} = 0$$

當坐標曲線互相垂直時 F=0,則

K= $\frac{eg - f^2}{EG - F^2} = -\frac{1}{\sqrt{EG}}\{\frac{\partial}{\partial u}(\frac{1}{\sqrt{E}}\frac{\partial \sqrt{G}}{\partial u}) + \frac{\partial}{\partial v}(\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v})\}$

曲面論基本定理

若 E、F、G、e、f、g 是給定的函數(of u,v),足夠多次可微且滿足 Gauss-Codazzi 方程,則存在一曲面 使得

$$I = Edu^2 + 2Fdudv + Gdv^2, II = edu^2 + 2fdudv + gdv^2$$

對實曲面 須 E>0,G>0,EG-F^2>0