The Four-or-More Vertex Theorem
Author(s): Robert Osserman
Source: *The American Mathematical Monthly*, Vol. 92, No. 5 (May, 1985), pp. 332-337
Published by: Taylor & Francis, Ltd. on behalf of the Mathematical Association of America
Stable URL: https://www.jstor.org/stable/2323126
Accessed: 27-05-2020 02:40 UTC

THE FOUR-OR-MORE VERTEX THEOREM

ROBERT OSSERMAN
Department of Mathematics, Stanford University, Stanford, CA 94305

The four-vertex theorem states that a smooth Jordan curve in the plane has at least four vertices. A *vertex* is a local maximum or minimum of the curvature. Thus, an ellipse has exactly four vertices, at the ends of the major and minor axes. This theorem is frequently proved, under the additional assumption that the curve is convex, in introductory differential geometry ([**2**], [**5**], [**6**], [**7**], [**13**], [**16**], [**21**]) as an early instance of a theorem requiring global rather than purely local arguments.

The four-vertex theorem (*Vierscheitelsatz*, *Théorème des quatre sommets*) has a long history, starting in 1909 with Mukhopadhaya [**18**], who stated and proved it for convex curves. There followed a succession of different proofs, generalizations, and analogies (see the References for a sample), including an interesting recent contribution due to Gluck [**9**], who proved a kind of converse. It is therefore somewhat surprising that the argument presented here seems not only to be new, but also to have a number of advantages over the usual proofs:

1. It makes immediately obvious geometrically why the result should be true.
2. It works not only for convex curves, but with only a little extra effort for arbitrary Jordan curves.
3. It is a direct proof, rather than the usual argument by contradiction. One consequence is that curves with *only* four vertices are seen to be special in certain ways; a large class of curves (even restricting to the convex case) must have six or more vertices.

The essence of the proof may be distilled in a single phrase: consider the circumscribed circle. In fact, one way to formulate the result would be the following.

THEOREM 1. *Let γ be a smooth (C^2) Jordan curve in the plane. Denote by C the circumscribed circle about γ. Then*
1. *$\gamma \cap C$ contains at least 2 points;*
2. *if $\gamma \cap C$ contains at least n points, then γ has at least $2n$ vertices.*

One could in fact make the second statement more precise:

THEOREM 1′. *In the notation of Theorem 1, if R is the radius of C, and if $\gamma \cap C$ contains at least n points, then either a whole arc of γ lies on C, or else γ has at least n vertices where the curvature κ satisfies $\kappa < 1/R$, and at least n vertices where $\kappa \geqslant 1/R$.*

We shall discuss at the end of this paper the question of the expected number of points on $\gamma \cap C$. Note that an immediate corollary of Theorem 1 is that whenever $\gamma \cap C$ contains an infinite number of points (as it well may), γ must have an infinite number of vertices.

The proof of Theorem 1 depends on three elementary and general geometric lemmas, as well as one lemma particular to the problem: Lemma 4 below.

LEMMA 1. *Let E be a compact set in the plane containing at least two points. Then among all circles C with the property that the closed disk bounded by C includes E, there is a unique one of minimum radius $R > 0$.*

Robert Osserman received his Ph.D. at Harvard University under the supervision of Lars Ahlfors. Since then he has been on the faculty of Stanford University, with temporary or visiting positions at the University of Colorado, New York University, Harvard, and the University of Warwick. During 1960–61 he was Head of the Mathematics Branch of the Office of Naval Research; he was a Fulbright Lecturer at the University of Paris (Orsay) in 1965–66 and held a Guggenheim Fellowship in 1976–77. He spent the past year (1983–84) at the Mathematical Sciences Research Institute in Berkeley. Mathematically, his interests have always been geometry-related, starting with geometric function theory and Riemann surfaces, through minimal surfaces to general differential geometry, isoperimetric inequalities, and most recently, smooth ergodic theory.

DEFINITION. The circle defined in Lemma 1 is called the *circumscribed circle* about E.

LEMMA 2. *If C is the circumscribed circle about E, then any arc of C greater than a semicircle must intersect E.*

Note. The proof of Lemma 2, as well as the uniqueness of C follow immediately from the observation that assuming the contrary, one could find a smaller circle enclosing E.

LEMMA 3. *Let a smooth oriented curve γ have the same unit tangent at a point P as a positively oriented circle C of radius R. Let κ be the curvature of γ. Then if $\kappa(P) > 1/R$, a neighborhood of P on γ lies inside C, while if $\kappa(P) < 1/R$, a neighborhood of P on γ lies outside C.*

We now derive Theorem 1 from these lemmas. Let γ be a Jordan curve, C the circumscribed circle, and R the radius of C. The first statement in Theorem 1 follows immediately from Lemma 2. To prove the second statement, let $P_1, \ldots, P_n$ be points of $\gamma \cap C$. If these points are ordered cyclically along γ, we obtain n arcs $\gamma_1, \ldots, \gamma_n$ of γ, each bounded by a pair of points on $\gamma \cap C$.

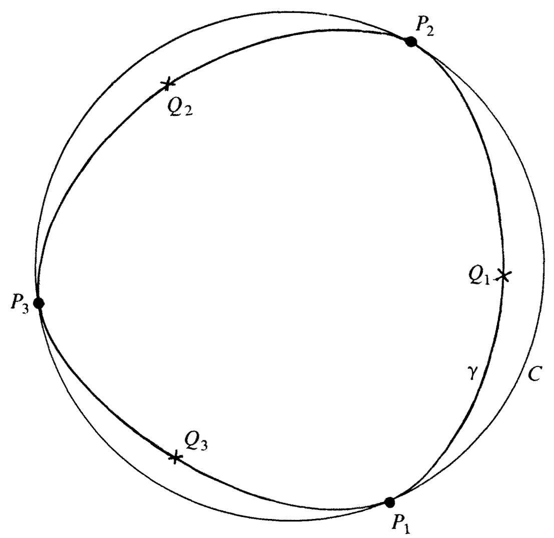

FIG. 1

Assertion. Each of the arcs γ_i either lies on C, or else contains a point Q_i such that the curvature κ of γ satisfies

$$\kappa(Q_i) < \frac{1}{R}. \tag{1}$$

Before proving this assertion, let us note why the theorem is an immediate consequence. First of all, we assume that γ and C both are positively oriented, so that the interior is to the left. Then at any point P_k of $\gamma \cap C$, the two curves have the same orientation and γ lies locally inside (or on) C. It follows from Lemma 3 that

$$\kappa(P_k) \geqslant \frac{1}{R}. \tag{2}$$

Since (2) holds at each endpoint of γ_i, it follows from (1) that κ has a minimum at some interior point Q_i' of γ_i, and that

$$\kappa(Q_i') < \frac{1}{R}. \tag{3}$$

We thus obtain n vertices satisfying (3). On the other hand, each arc γ_k' of γ between successive Q_i contains at least one point P_k of $\gamma \cap C$. In view of (1) and (2), there is an interior point P_k' of

γ_k' where κ is a maximum, and

(4) $$\kappa(P_k') \geqslant \frac{1}{R}.$$

We thus get n more vertices, thereby proving Theorem 1′, and hence Theorem 1. (We have ignored the possibility that one of the γ_i lies on C, in which case every point of γ_i is trivially a vertex.)

It remains to prove the Assertion above. We formulate it as a separate lemma.

LEMMA 4. *Let γ be a positively oriented Jordan curve, C the circumscribed circle and P_1, P_2 points of $\gamma \cap C$. Let γ_1 be the (positively oriented) arc of γ from P_1 to P_2. Then either γ_1 coincides with the circular arc P_1P_2 or else there is a point Q_1 on C satisfying* (1), *where R is the radius of C.*

Proof. By Lemma 2 we may assume that the positively oriented arc of C from P_1 to P_2 is included in a closed semicircle; if not, by Lemma 2, there is a point P_2' between P_1 and P_2 such that the arc of C from P_1 to P_2' does lie in a (closed) semicircle, and we may apply the argument below to the subarc γ_1' of γ_1 from P_1 to P_2'. The corresponding point Q_1 of γ_1' satisfying (1) will also lie on γ_1.

For convenience of referral, assume that C is centered at the origin, and that P_1, P_2 lie on the same vertical line in the right half-plane, with P_2 above P_1 (Fig. 2).

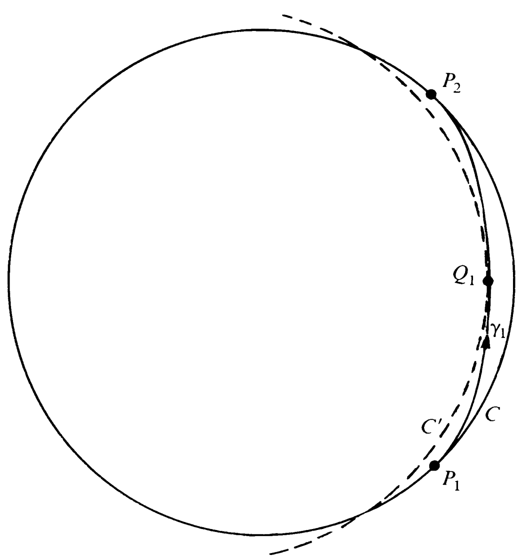

FIG. 2

There are two possibilities. Either γ_1 coincides with the circular arc P_1P_2, or else there is some point Q on γ_1 that lies strictly inside C. Consider first the case where γ is convex. If we translate the circle determined by P_1, Q, P_2 to the left, there will be a last moment at which it intersects γ_1. Let C' be the corresponding position of the circle, and let Q_1 be a point of the intersection $C' \cap \gamma_1$. Since the radius R' of C' satisfies $R' > R$, and since γ_1 lies locally outside C' at Q_1, it follows from Lemma 3 that

$$\kappa(Q_1) \leqslant \frac{1}{R'} < \frac{1}{R}.$$

This proves the lemma, and hence the theorem, for the case of convex curves.

Precisely the same argument holds for general Jordan curves, with one additional *caveat*: we must use the Jordan property to guarantee that γ_1 has the same orientation as C' at Q_1. (In fact, for non-Jordan curves that need not be the case, and the lemma, as well as the theorem, need not hold; see Fig. 3.)

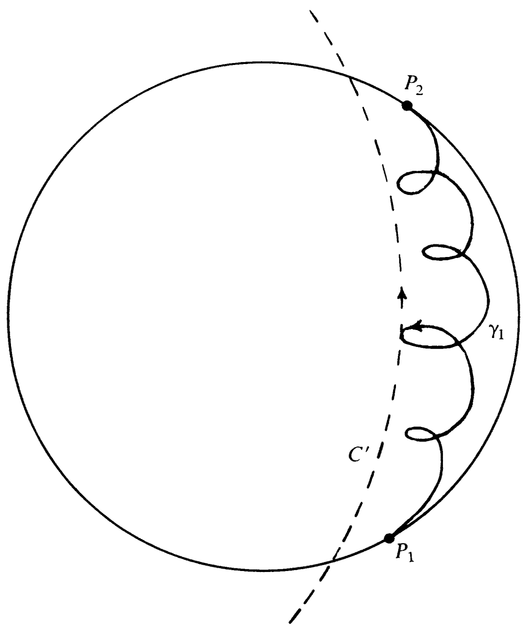

FIG. 3

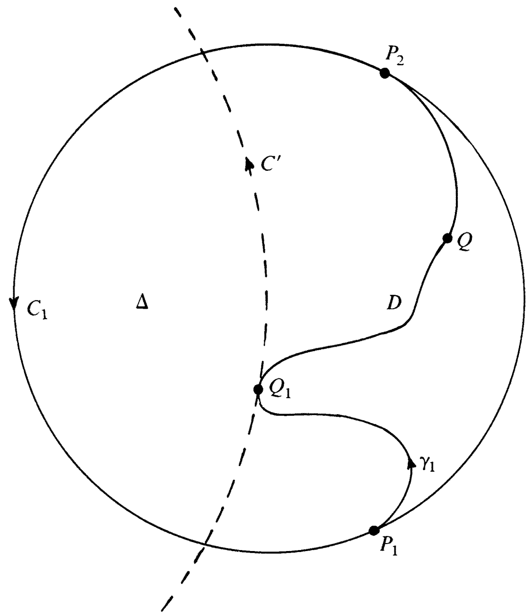

FIG. 4

Under the assumption that γ_1 has no self-intersections, the closed curve, consisting of γ_1 followed by the arc C_1 of C going in the positive direction from P_2 to P_1, is a Jordan curve whose interior is a domain D included in the interior of C. Note that the positive orientation induced on γ_1 as boundary of D coincides with its original orientation as part of γ, since at the points P_1 and P_2, both coincide with the positive orientation of C. Once again, there are two cases to consider. Either γ_1 coincides with the arc of C from P_1 to P_2, or else γ_1 contains a point Q strictly inside C. In the latter case, we may choose Q to the right of the vertical line through P_1 and P_2. (See also Remark 1 following the proof.) Then the circle determined by P_1QP_2 has radius $R' > R$. Translating this circle to the left, we again find a circle C' containing a point Q_1 of γ_1 such that all further translates of C' to the left fail to intersect γ_1. (See Fig. 4.) It follows that the interiors of C and C' intersect in a domain Δ that is included in D. Thus, both γ_1 and C' have the same orientation at Q_1, and we may apply Lemma 2 as before to deduce

$$\kappa(Q_1) \leqslant \frac{1}{R'} < \frac{1}{R}.$$

This proves Lemma 4 and Theorem 1 for arbitrary Jordan curves.

REMARK 1. A slight modification of the argument above produces sharper quantitative results. Consider all circular arcs from P_1 to P_2 lying inside C. Let C'' be the one farthest to the left intersecting γ_1, and let Q'' be a point of $\gamma_1 \cap C''$. There are three cases, depending on whether Q'' is to the right, to the left, or on the vertical line P_1P_2. In the last case, the argument above shows that $\kappa(Q'') \leqslant 0$. In the other two cases, C'' is a proper circle of radius R''. If Q'' is to the right of the line P_1P_2, then again

$$\kappa(Q'') \leqslant \frac{1}{R''} < \frac{1}{R}.$$

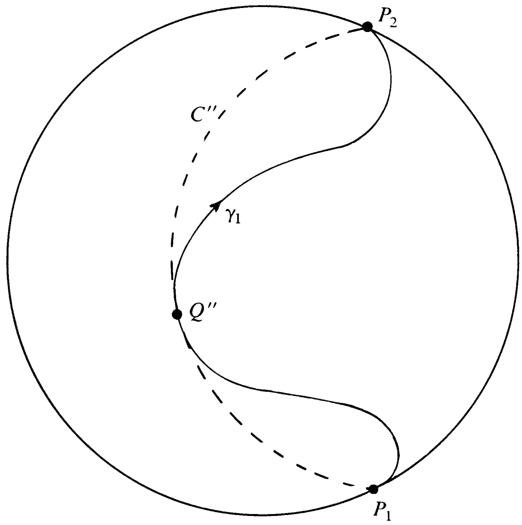

FIG. 5

If Q'' is the left, one has the stronger result that

$$\kappa(Q'') \leqslant -\frac{1}{R''} < 0.$$

For this last, one notes that at Q'', the positive orientation of γ_1 coincides with the negative orientation of C'' (Fig. 5).

REMARK 2. It might seem natural to carry out a basically equivalent "dual" approach, using inscribed, rather than circumscribed circles. On closer examination, however, the use of inscribed circles is considerably less straightforward. In fact, even their definition requires some care, and they are generally not unique. A paper of Jackson [**12**] contains a proof of the four-vertex theorem along those lines. He uses a proof by Erdös of the existence of specially adapted inscribed circles (p. 568). He also proves a result (Lemma 4.1) more general than our Lemma 4, making use of the Gauss-Bonnet theorem.

REMARK 3. As we said at the outset, the usual proofs of the four-vertex theorem show that the presence of fewer than four vertices would lead to a contradiction. Such proofs give no hint as to the actual number of vertices present, either on a given curve, or "in general". It follows from Theorem 1 that a curve with only four vertices must intersect its circumscribed circle in only two points. By Lemma 2, those two points must be antipodal points of the circle. Clearly, that is a fairly special property, even within the class of convex curves. Further properties that must be satisfied by curves with only four vertices have been derived by Jackson [**12**] and others. We are thus led to two questions, each of which may be considered either for the class of smooth convex curves, or more generally, for closed Jordan curves.

1. Is it more likely for a curve to have only four vertices, or to have at least six vertices?
2. Is it more likely for a curve to intersect its circumscribed circle in only two points, or in at least three points?

Intuitively, one may expect a "tripod" effect; that is, the circumscribed circle is most likely to touch the curve at *exactly* three points (see Fig. 1). We are thus led to formulate the following precise problem.

Is there a natural measure on the space of all smooth closed curves (*either convex or Jordan*)? *In terms of such a measure*, *what are the relative sizes of the sets of curves which intersect their circumscribed circles in* (a) *exactly two points*, (b) *exactly three points*, (c) *more than three points*?

Note added in proof: I should like to thank David Gale, David Hoffman, and Erwin Lutwak for interesting comments and references concerning the problem posed at the end of this paper. In particular, a paper of Tudor Zamfirescu (Proc. Amer. Math. Soc., 80, 3 (1980) 455–457) studies the number of contact points of a convex curve with its circumscribed circle and shows that in the sense of Baire categories, "most" convex curves have exactly three contact points. On the other hand, an earlier paper of Peter Gruber (Math. Ann., 229 (1977) 259–266) shows that in the same sense, "most" convex curves are not C^2. Thus, the Baire category approach is not appropriate to the four-vertex problem. Similarly, it is easy to see that in the C^1-topology, the set of convex curves with three points of contact is dense in the set of C^1 convex curves. However, in the C^2-topology there is an open set of convex curves with just two points of contact. In fact, any convex curve that contacts its circumscribed circle in exactly two antipodal points, and whose curvature at those points is strictly greater than that of the circumscribed circle, has a neighborhood (in the C^2-topology) with the same property. Thus, the problem of finding an appropriate measure for this question is a subtle one.

References

1. M. Barner and F. Flohr, Der Vierscheitelsatz und seine Verallgemeinerung, Der Mathematikunterricht, 4 (1958) 43–73.

2. Marcel Berger and Bernard Costiaux, Géométrie Différentielle, Armand Colin, Paris, 1972 (see p. 362).

3. Wilhelm Blaschke, Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve, Rend. Circ. Mat. Palermo, 36 (1913) 220–222.

4. ______, Kreis und Kugel, Leipzig, 1916.

5. ______, Vorlesungen über die Differentialgeometrie I, 3. Aufl. Springer-Verlag, Berlin, 1930 (see p. 31).

6. S. S. Chern, Curves and surfaces in euclidean space, MAA Studies in Math. 4: Studies in Global Geometry and Analysis, Mathematical Association of America, Washington, DC, 1967, 16–56 (see p. 23).

7. Manfredo P. do Carmo, Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, NJ, 1976 (see p. 37).

8. David Fog, Über den Vierscheitelsatz und seine Verallgemeinerungen, Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl., Berlin, 1933, 251–254.

9. Herman Gluck, The converse to the four-vertex theorem, L'Enseignement Math., 17 (1971) 295–309.

10. W. C. Graustein, Extensions of the four-vertex theorem, Trans. Amer. Math. Soc., 41 (1937) 9–23.

11. Otto Haupt, Zur Theorie der Ordnung reeller Kurven in der Ebene bezüglich vorgegebener Kurvenscharen, Monatsh. Math. Phys., 40 (1933) 1–53.

12. S. B. Jackson, Vertices of plane curves, Bull. Amer. Math. Soc., 50 (1944) 564–578.

13. Wilhelm Klingenberg, A Course in Differential Geometry, Springer-Verlag, New York, 1978 (see p. 28).

14. A. Kneser, Bemerkungen über die Anzahl der Extrema der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nicht-euklidischen Geometrie, Festschrift Heinrich Weber, Teubner 1912, 170–180.

15. H. Kneser, Neuer Beweis des Vierscheitelsatzes, Christian Huygens, 2 (1922/23) 315–318.

16. Detlef Laugwitz, Differential and Riemannian Geometry, Academic Press, New York, 1965 (see p. 201).

17. Hans Mohrmann, Die Minimalzahl der Scheitel einer geschlossenen konvexen Kurve, Rend. Circ. Mat. Palermo, 37 (1914) 267–268.

18. S. Mukhopadhaya, New methods in the geometry of a plane arc, Bull. Calcutta Math. Soc., 1 (1909) 31–37.

19. ______, Extended minimum-number theorems of cyclic and sextactic points on a plane convex oval, Math. Z., 33 (1931) 648–662.

20. Peter Scherk, The four-vertex theorem, Proc. First Canadian Math. Congress, Montreal, 1945. University of Toronto Press, Toronto, 1946, 97–102.

21. Dirk J. Struik, Lectures on Classical Differential Geometry, 2nd ed., Addison-Wesley, Reading, MA, 1961 (see p. 48).

22. L. Vietoris, Ein einfacher Beweis des Vierscheitelsatzes der ebenen Kurven, Arch. Math., 3 (1952) 304–306.

23. Wolfgang Vogt, Über monotongekrümmte Kurven, J. Reine Angew. Math., 144 (1914) 239–248.

ANSWER TO PHOTO ON PAGE 327

Arthur Sard (1909–1980). The picture was taken in 1964.